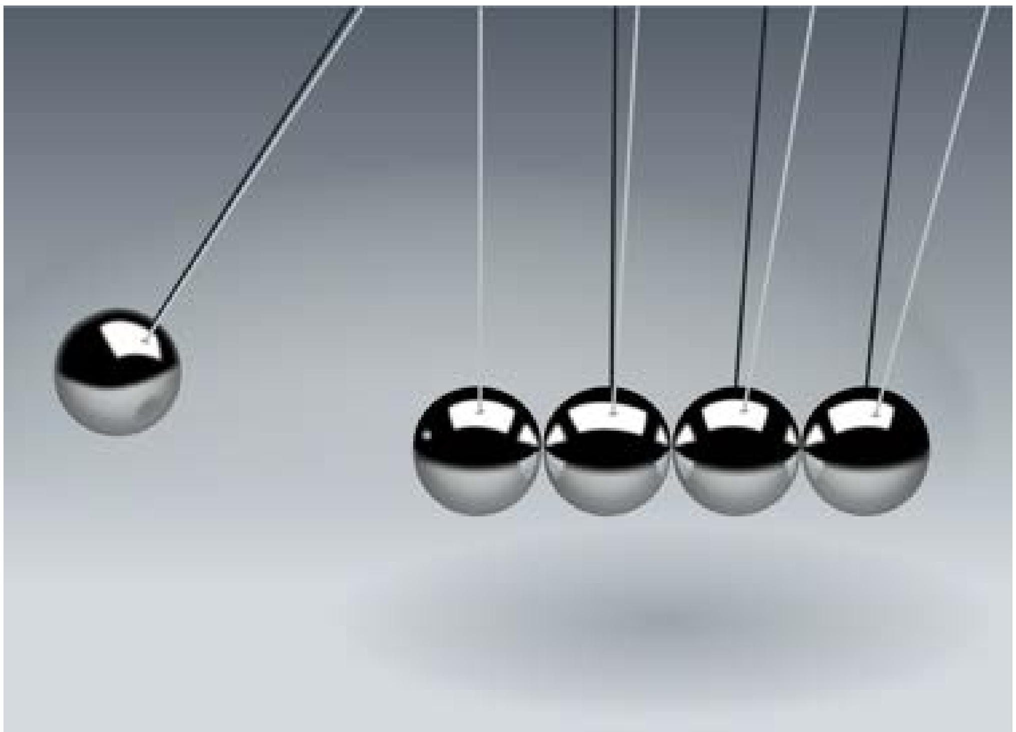
Physics Problems



Workbook 2: Solutions

To help with preparation for the
Physics Aptitude Test (PAT) at the
University of Oxford

Department of Engineering Science
Department of Physics
Oxford Materials

Contents

Chapter 1

Warm Up Problems

1. **Problem**
   It takes 20 minutes to fill a bath tub by running the hot water tap. It takes 15 minutes to fill the same bath tub by running the cold water tap. It takes 10 minutes to drain the bath tub by removing the plug.

   If both taps are running and the plug is removed, how long will it take to fill the bath tub?

   Hint
   This is a very similar question to the problem about Hayley and Rob painting a house from Workbook 1, so any techniques used to tackle that question will come in handy here. In particular, a rate of flow problem such as this is very similar to combining resistors in parallel.

   Solution 1
   Recognising that this is a rate problem, the rate at which the hot water tap fills up the bath tub is 1/20; the cold water tap rate is 1/15; and removing the plug drains the bath at a rate of 1/10. Therefore, when both taps are running and the plug is removed:

$$\text{rate} = \frac{1}{\text{time}} = \frac{1}{20} + \frac{1}{15} - \frac{1}{10} \tag{1.1}$$

$$= \frac{3}{60} + \frac{4}{60} - \frac{6}{60} \tag{1.2}$$

$$= \frac{1}{60} \tag{1.3}$$

   This means that the bath takes 60 minutes to fill.

   Solution 2
   Slightly more rigorously: let the bath tub have a volume V. Let:

   - h be the rate at which hot water flows into the bath

- c be the rate at which cold water flows into the bath
- d be the rate at which water is drained from the bath

Note that h, c and d will all have units of volume of water per unit time. From the information given in the question, this means that

$$20h = 15c = 10d = V \tag{1.4}$$

and so $h = V/20$, $c = V/15$ and $d = V/10$. If both taps are running and the plug is removed, this corresponds to a rate of $h + c - d$, which means that the bath tub fills up at a rate

$$h + c - d = \frac{V}{20} + \frac{V}{15} - \frac{V}{10} = \frac{V}{60} \tag{1.5}$$

Rearranging this equation gives

$$60(h + c - d) = V \tag{1.6}$$

and so it takes 60 minutes for the bath tub to fill up.

2. **Problem**
Find the height of the triangle in Figure 1.1 if $a = 9\,\text{cm}$, $b = 16\,\text{cm}$ and $c = 25\,\text{cm}$. Note: the triangle is not drawn to scale!

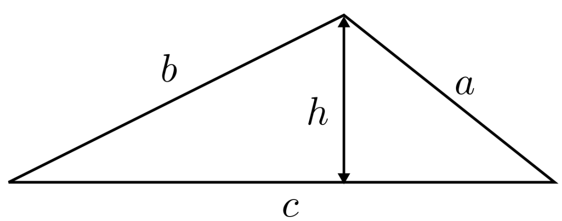


Figure 1.1: A triangle with lengths $a = 9\,\text{cm}$, $b = 16\,\text{cm}$ and $c = 25\,\text{cm}$.

Hint
In theory, this problem could be solved by a primary school student! Do not be tricked into recognising a 3-4-5 Pythagorean triple.

Solution
Since $a + b = c$, the height is 0. If you were to try to draw the triangle, it would end up being a straight line.

3. **Problem**
A family exercise their dog in the following way:

- The parents sit on a bench while the children walk to a rock a distance D away, calling to the dog as they walk.
- The parents and the children alternately call the dog, who runs from one group to the other and back again.

- The children then walk back to their parents, still calling the dog on their way back.

If the dog travels at a constant speed v_d and the children at a constant speed v_c (with $v_c < v_d$), calculate how far the dog runs in total.

Hint
Thinking back to the question about the pirate, ninja and parrot from Workbook 1, there is a similarly quick way to solve this – do not try to break it down into too many steps. Nothing more than GCSE Physics and clear thinking is required. Note that all the speeds are constant.

Solution
The dog runs at a constant speed for the time it takes the children to walk to the rock and back again. This time is given by the total distance travelled divided by their speed, or $2D/v_c$. Hence the distance that the dog travels is:

$$\text{distance} = \text{speed} \times \text{time} = v_d \times \frac{2D}{v_c} = 2D\frac{v_d}{v_c} \quad (1.7)$$

4. **Problem**
There are 3 cards hidden under a cloth on a table. It is known that one card is white on both sides, one is black on both sides and the other is black on one side and white on the other.

I select a card at random and its upper face is white. What are the odds that its other side is also white?

Hint
If you're familiar with the classic Monty Hall problem, similar thinking should help here. You may think that you have an equal chance of picking any of the three cards – this was true, but looking does more than just eliminate the black card. It may help to label each side from 1 to 6 and assign odds that way.

Solution
Let the white-white card have sides W_1 and W_2, the black-black card have sides B_1 and B_2 and the white-black card have sides W_3 and B_3. With no knowledge other than the fact that I have picked a white side, the side that I have picked could either be W_1, W_2 or W_3:

- If it is W_1 then the other side is also white
- If it is W_2 then the other side is also white
- If it is W_3 then the other side is black

So the probability that the other side is also white is 2/3.

There are a couple of other ways of thinking about the problem:

- It is more likely that I have selected the white-white card, since half the time the white-black card will have its black face up

- Consider running the game twice, with all cards flipped the second time around. Removing the cloth, there are three white cards. It is equally likely that you have selected either of these. Of these, two are the white-white card. So there is a $2/3$ chance that the other side is also white.

5. **Problem**
   Imagine that you have a height h and are standing at a distance d from a mirror, looking at your own reflection. In order to be able to see a full-length view of yourself, the minimum size of the plane mirror must be:

   (a) $h/4$

   (b) $h/2$

   (c) $3h/4$

   (d) h

   (e) Depends on the exact value of d

   Hint
   Draw a ray diagram. How can you see both your head and your feet in a mirror? Remember the law of reflection.

   Solution
   First we will consider what happens when I look at my own reflection in a mirror of height h. Figure 1.2 shows a person of height h looking at their own reflection in a mirror of height h.

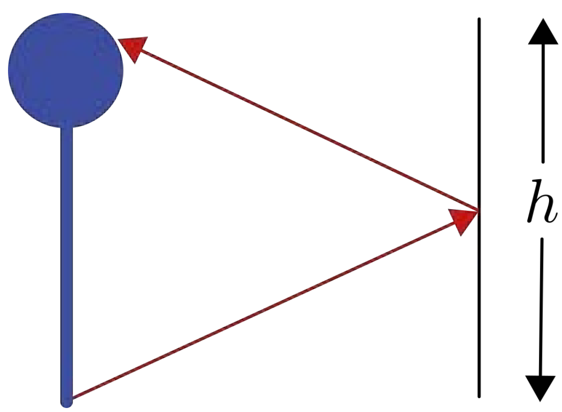


Figure 1.2: A person of height h looking at their reflection in a mirror of height h.

In order for me to see a body part, light rays have to hit the body part, reflect off the mirror and reach my eyes. The limiting factor are the rays from my feet, which reflect off the mirror at a distance halfway up the mirror and then reach my eyes. Therefore, every part of the mirror underneath this point is redundant. The height of the mirror is therefore $h/2$ and is independent of the distance d I am away from it.

6. **Problem**

   Zoe wishes to 'spear' a fish with a laser – that is, she wants to shine a laser onto a lake and have the light beam hit a fish below the surface of the water. Should she aim the laser beam above, below, or directly at the observed fish to make a direct hit?

   Hint

   First think about where Zoe would need to aim if she were throwing a physical spear into the water and wanted to hit the fish directly. How does shining a laser beam affect the physics involved?

   Solution

   Figure 1.3 shows a diagram of the situation.

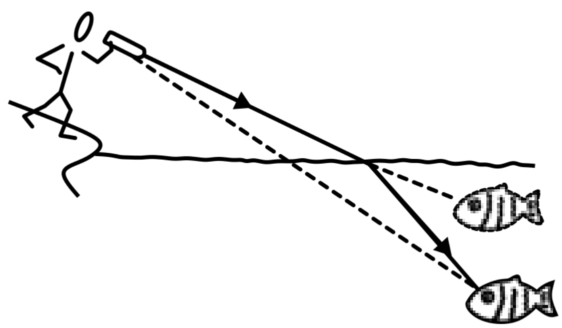


Figure 1.3: A diagram of Zoe aiming her laser at the fish.

To Zoe, the fish appears slightly above where it physically is in the water due to the refraction of light. If Zoe were throwing a physical spear, she would have to compensate for this refraction of light and aim below the observed fish. But since the 'spear' in this question is an actual beam of light, she should aim directly at the fish. The laser will refract towards the fish in exactly the same way that light bouncing off the fish refracts towards Zoe's eyes.

Chapter 2

Circular Motion

2.1 Introductory Problems

1. **Problem**
   The faster I swing a pendulum around my head, the closer the string gets to being perfectly horizontal.

   (a) With the aid of a clear diagram, explain why this is the case.

   (b) How fast must the pendulum mass be travelling for the string to be exactly horizontal?

   Hint
   Draw a free body diagram to show the forces acting on the mass. There are only two, but you also know the direction of the resultant force.

   Solution

   (a) Consider Figure 2.1.
   The tension in the string provides the centripetal force required to maintain circular motion. Resolving forces horizontally:

   $$T\cos\theta = \frac{mv^2}{r} \tag{2.1}$$

   where mv^2/r is the centripetal force. Likewise, to maintain vertical equilibrium we require that

   $$T\sin\theta = mg \tag{2.2}$$

   Taking equation 2.2 and dividing it by equation 2.1 then yields

   $$\tan\theta = \frac{rg}{v^2} \tag{2.3}$$

   As v increases, $\tan\theta$ decreases, meaning that θ also decreases. Since a decrease in θ corresponds to the string becoming more horizontal,

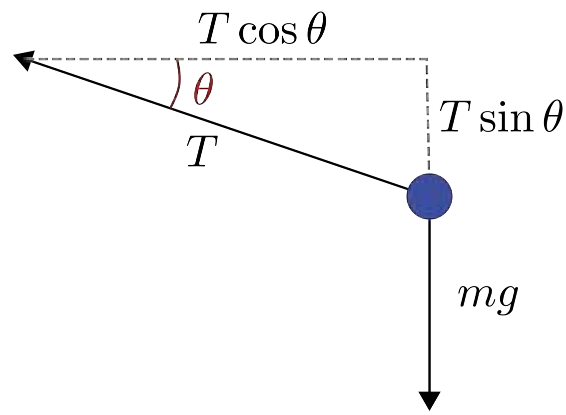


Figure 2.1: A diagram showing the forces acting on the pendulum mass.

the faster I swing a pendulum around my head, the closer the string gets to being perfectly horizontal.

(b) For the string to be exactly horizontal, we require that $\theta = 0$. For non-zero r and g this means the pendulum mass must be travelling infinitely fast! This can also be seen from the diagram: there must always be a vertical component of tension that is equal to the weight, and so there must always be an angle $\theta \neq 0$.

2. **Problem**

Assume that the Earth is a perfect sphere of radius 6400 km, spinning on its axis. When a person stands on some weighing scales at the North Pole, the scales read 800 N.

We will now think about what would happen if the person were to stand on the scales at different points around the globe.

(a) First qualitatively: if the person were to weigh themselves at the equator, would the reading on the scales be higher, lower, or the same value?

(b) Now quantitatively: calculate the difference in the reading on the scales if the person were to weigh themselves at the equator compared to the reading at the North Pole.

(c) What would the reading on the scales be if the person were to weigh themselves in Oxford, which has a latitude of 51.8° North?

Hint

(a) Recall that scales work by measuring the contact force that they provide on the person standing on them. Draw a free body diagram for the person. Consider whether or not the person is accelerating in each scenario and use Newton's laws.

(b) Remember that the centripetal force is the resultant force – it is always provided by something else.

(c) The equator is at a latitude of $0°$ and the North Pole is at a latitude of $90°$. Have you drawn a cross-section of the Earth and included a right-angled triangle? In which direction does your weight and contact force act? Do you need an additional force? What would happen if you were on an ice-rink?

Solution

(a) There is no centripetal force acting on a person standing at the poles, but there is such a centripetal force at the equator. This is the resultant force and points radially inwards.

The only two forces acting on the person are their weight $W = mg$ and the normal reaction force N.[1] Whilst these two balance when the person stands at the pole, when at the equator the normal reaction force must be less than the weight, such that the resultant force provides the inwards centripetal force.

By Newton's third law, the normal reaction force is provided by the scales and hence the scales are pushed down with this same force. Therefore the reading would be lower at the equator than at the poles.

(b) Since the resultant force provides the centripetal force:

$$W - N = \frac{mv^2}{r} \tag{2.4}$$

Using the facts that $v = \omega r$ and $\omega = 2\pi/T$, where T is the period of circular motion, this means that the normal reaction force at the equator is

$$N = W - \frac{m}{r}(\omega r)^2 \tag{2.5}$$

$$= W - mr \times \left(\frac{2\pi}{T}\right)^2 \tag{2.6}$$

$$= W - \frac{Wr}{g} \times \frac{4\pi^2}{T^2} \tag{2.7}$$

where we have substituted $m = W/g$ for the mass of the person. This is to be contrasted to the normal reaction force at the pole, which is just $N = W = mg$ due to the lack of any centripetal force. Hence the difference between the two readings ΔN is

$$\Delta N = \frac{Wr}{g} \times \frac{4\pi^2}{T^2} \tag{2.8}$$

[1] Note that this is only the case at the pole and the equator. At any other latitude there will also be a frictional force – see the solution to part (c) for a fuller explanation.

The period T of the circular motion is just the rotation period of the Earth: 24 hours or 86400 seconds. $W = 800\,\mathrm{N}$, $g = 9.81\,\mathrm{ms}^{-2}$ and $r = 6400\,\mathrm{km}$ is the radius of the Earth. Plugging all of this in:

$$\Delta N = \frac{800 \times 6.4 \times 10^6}{9.81} \times 4\pi^2 86400^2 \approx 2.8\,\mathrm{N} \tag{2.9}$$

(c) Consider Figure 2.2.

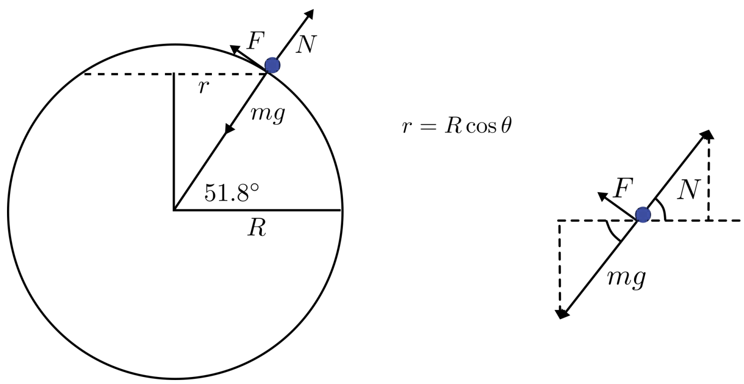


Figure 2.2: A diagram showing the forces acting on the person in Oxford.

Note that the radius of circular motion is now $r = R\cos\theta$ where R is the radius of the Earth, since the person standing at 51.8° North travels a smaller circle in 24 hours due to the Earth's rotation than a person standing at the equator.

The centripetal force acts along this distance r. However, the forces mg and N act along the same line (radially inwards and outwards towards and away from the centre of the Earth), so there is no way to resolve these two forces in order to provide a centripetal force in the correct direction. This means that there must exist a frictional force F, as indicated in Figure 2.2, in order for there to be an inwards-pointing centripetal force.

Resolving forces vertically:

$$N\sin\theta + F\cos\theta = mg\sin\theta \tag{2.10}$$

where $\theta = 51.8°$ is the latitude of the person. Rearranging:

$$F = (mg - N)\tan\theta \tag{2.11}$$

Resolving forces horizontally:

$$\frac{mv^2}{r} = m\omega^2 r = mg\cos\theta + F\sin\theta - N\cos\theta \tag{2.12}$$

Substituting $r = R\cos\theta$ and dividing through by $\cos\theta$ gives

$$m\omega^2 R = mg + F\tan\theta - N \tag{2.13}$$

Substituting equation 2.11 into equation 2.13 gives

$$m\omega^2 R = mg + (mg - N)\tan^2\theta - N \tag{2.14}$$

Rearranging:

$$m\omega^2 R = mg(\tan^2\theta + 1) - N(\tan^2\theta + 1) \tag{2.15}$$

and so

$$N = mg - \frac{m\omega^2 R}{(\tan^2\theta + 1)} \tag{2.16}$$

$$= mg - m\omega^2 R\cos^2\theta \tag{2.17}$$

where we have used the fact that

$$\tan^2\theta + 1 = \sec^2\theta = \frac{1}{\cos^2\theta} \tag{2.18}$$

Replacing m with W/g and once again using the fact that $\omega = 2\pi/T$, equation 2.17 becomes

$$N = W - \frac{W}{g} \times \frac{4\pi^2}{T^2} \times R\cos^2\theta \tag{2.19}$$

$$= 800 - \frac{800}{9.81} \times \frac{4\pi^2}{86400^2} \times 6.4 \times 10^6 \times \cos^2(51.8°) \tag{2.20}$$

$$\approx 798.9\,\mathrm{N} \tag{2.21}$$

3. **Problem**
The Earth is actually an oblate spheroid – that is, its equatorial diameter is larger than its North-to-South diameter.

   (a) How would this affect the person's weight at the equator and at the poles?

   (b) Suggest why the Earth is this shape.

   Hint

   (a) It is helpful to know Newton's law of gravitation:

$$F = \frac{GMm}{r^2} \tag{2.22}$$

   How does the gravitational field strength vary with distance from the centre of mass?

(b) Have you ever seen a chef making a pizza base? Similar physics is involved here.

Solution

(a) There are two factors to consider here:

i. If the person were at the equator, they would be further away from the centre of the Earth (which is the centre of mass of the Earth) than if they were at the poles. This means that the force of gravity the person would experience at the equator would be weaker, since

$$mg = \frac{GMm}{r^2} \tag{2.23}$$

ii. If the person were at the equator, the larger radius would mean that there must be a larger centripetal force to keep the period the same, since

$$F_c = m\omega^2 r \tag{2.24}$$

where F_c is the centripetal force.

Both of these effects would decrease the weight recorded on the mass balance at the equator. However, the first effect would physically decrease the person's weight.

(b) The Earth is an oblate spheroid and bulges out along the equator due to its rotation. The centrifugal force due to this rotation causes the Earth to have this shape.

You may have heard that the centrifugal force is a fictitious force. This is true, but it all depends on your reference frame.

In an inertial frame of reference (that is, a frame of reference which is not accelerating), fictitious forces such as the centrifugal and Coriolis forces don't exist. However, the surface of the Earth is *not* an inertial frame of reference, because it is continuously accelerating as it rotates around the Earth's centre. In this situation, it makes sense to talk about centrifugal forces.

If you were spinning quickly on a carousel in a playground, you would feel a force (the centrifugal force) pushing you outwards away from the carousel's centre, and this force would become stronger the faster the carousel rotated. In a similar way, the Earth's rotation causes its equator to bulge out more than the poles, since the equator spins faster than the poles.

2.2 Further Problems

4. **Problem**
Determine the length of a day in which a person standing on the equator would appear weightless.

Hint
What is the condition on the contact force that leads to the experience of weightlessness? How is the resultant force dependent on the period?

Solution
From a previous problem, we know that it is the difference between the weight and the normal reaction force (i.e. the resultant force) that provides the centripetal force:

$$mg - N = m\omega^2 r = \frac{4\pi^2 mr}{T^2} \tag{2.25}$$

where we have substituted $\omega = 2\pi/T$. The condition for weightlessness is for the normal reaction force to vanish, so setting $N = 0$ and rearranging for T:

$$T = 2\pi\sqrt{\frac{r}{g}} \tag{2.26}$$

$$= 2\pi\sqrt{\frac{6.4 \times 10^6}{9.81}} \tag{2.27}$$

$$\approx 1.4\,\text{hours} \tag{2.28}$$

5. **Problem**
Newton's cannon is a thought-experiment whereby a cannonball is fired horizontally from a high mountain top at varying speeds. If the cannonball is fired at or above some critical velocity v, the surface of the Earth will curve away faster than the ball falls back to Earth – the cannonball would now be in orbit.

(a) Determine the orbital velocity. You may assume its orbital radius is 6400 km and ignore air resistance.

(b) Hence, or otherwise, determine the period of the orbit.

Hint
If there is only one force acting on an object in a circular orbit, this force must also be the centripetal force.

Solution

(a) The only force acting on the ball is gravity, so this provides the centripetal force:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \tag{2.29}$$

where m is the mass of the cannonball and M is the mass of the Earth. Rearranging for the velocity yields

$$v = \sqrt{\frac{GM}{r}} \tag{2.30}$$

If we don't know what the mass of the Earth (6.0×10^{24} kg for future reference) we can substitute in the definition of the gravitational field strength

$$g = \frac{GM}{r^2} \tag{2.31}$$

to get

$$v = \sqrt{gr} = \sqrt{9.81 \times 6.4 \times 10^6} \approx 7.9\,\mathrm{kms}^{-1} \tag{2.32}$$

(b) We know that $v = \omega r$ and $\omega = 2\pi/T$. This means that

$$T = \frac{2\pi r}{v} = \frac{2\pi \times 6.4 \times 10^6}{7.9 \times 10^3} \approx 1.4\,\text{hours} \tag{2.33}$$

Food for thought: is it a coincidence that this is the same length of time as the length a day would have to be for a person standing on the equator to appear weightless? If not, why not?

6. **Problem**
A penny dropped from the top of the Burj Khalifa (height 828 m) in Dubai (latitude 25° North) will miss a target directly below it. Why? By what distance will the penny miss the target?

Hint
This is a synoptic question – circular motion and...what else is involved? Can you work out the difference in linear speed between the top and bottom of the building?

Solution
At the top of the Burj Khalifa you are travelling in a (slightly) larger circle around the globe as the Earth rotates once every 24 hours than if you were at the bottom of the building. Hence, people at the top of the Burj Khalifa are travelling faster.

The size of this circle which you travel around the globe depends on your latitude, as shown in Figure 2.3. Whilst at the equator you travel a circle of radius R, at a latitude of θ you travel a circle of radius $R\cos\theta$. At the poles, where $\theta = 90°$, you don't travel at all since the poles are not rotating!

Ignoring air resistance, a penny which falls from the top of the Burj Khalifa to the bottom has an initial horizontal velocity relative to the ground, and so will miss a target placed directly below it.

The linear velocity at the top of the building is given by

$$v_{top} = \frac{2\pi(R+h)\cos\theta}{T} \tag{2.34}$$

where R is the radius of the Earth, h is the height of the Burj Khalifa, T is one rotation period of the Earth and θ is the latitude.

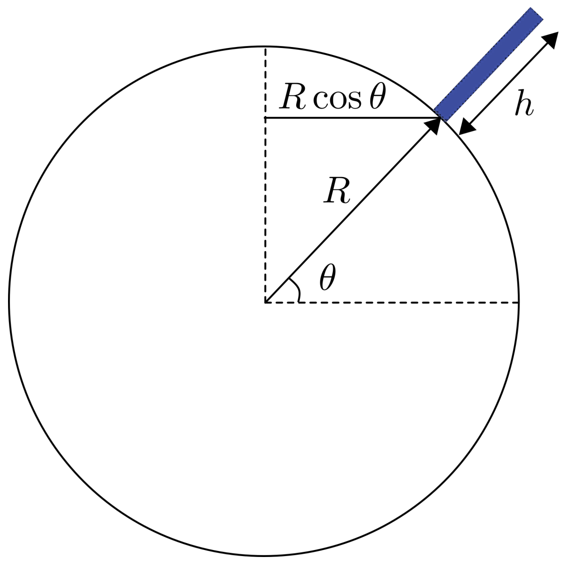


Figure 2.3: A diagram (not to scale!) showing the Burj Khalifa on the Earth. An object's latitude θ determines the radius $R\cos\theta$ of its circular motion due to the Earth's rotation.

The linear velocity at the bottom of the building is given by

$$v_{base} = \frac{2\pi R \cos\theta}{T} \tag{2.35}$$

This means that the relative velocity between the top and bottom of the building is

$$\Delta v = v_{top} - v_{base} = \frac{2\pi h \cos\theta}{T} \tag{2.36}$$

$$= \frac{2\pi \times 828 \times \cos 25^\circ}{86400} \approx 5.457\,\mathrm{cms}^{-1} \tag{2.37}$$

In other words, the penny is always travelling approximately $5.5\,\mathrm{cms}^{-1}$ faster horizontally than the target on the ground.

Assuming g to be constant, we can calculate the time it takes for the penny to fall using a standard SUVAT equation. In the vertical direction:

$$s = ut + \frac{1}{2}at^2 \tag{2.38}$$

The penny is dropped with an initial vertical velocity $u = 0$ and so

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 828}{9.81}} \approx 13\,\mathrm{s} \tag{2.39}$$

Finally, using distance = speed × time, the penny will be

$$5.457 \times 13 \approx 70.9\,\text{cm} \tag{2.40}$$

ahead of its mark.

2.3 Extension Problems

7. **Problem**

   A smooth marble is initially at rest at the top of a much larger smooth hemisphere of radius r. The marble is given a slight nudge and begins to slide down the hemisphere.

   (a) At what angle from the vertical will the marble leave the surface of the hemisphere?

   (b) How far away from the base will the marble land?

   Hint

   (a) What do you know about the contact force at the point of interest?

   (b) Have you resolved your initial velocity into useful components? The marble in a fishbowl question from Workbook 1 may be a useful guide.

   Solution

   (a) Figure 2.4 illustrates the problem.

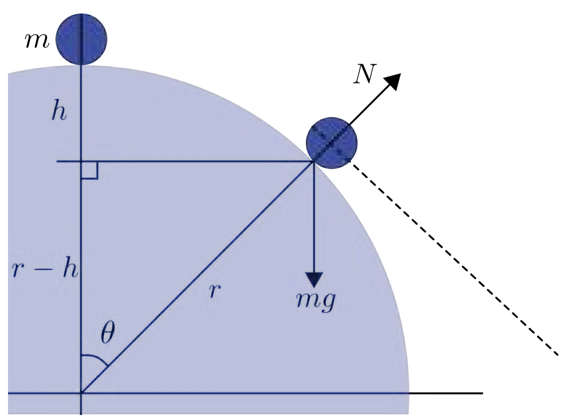


Figure 2.4: A smooth marble sliding down a smooth hemisphere.

Although the velocity of the marble is not constant, since it follows a circular path as it rolls down the hemisphere, there must exist a centripetal force at any instant. Balancing forces horizontally:

$$mg\cos\theta - N = \frac{mv^2}{r} \tag{2.41}$$

Now we can apply the conservation of energy. If we take the bottom of the hemisphere to be our level of zero gravitational potential energy, then the marble starts off with no kinetic energy and a potential energy equal to mgr, since it starts off a perpendicular distance r above the ground. By the conservation of energy, the marble must *always* have a total energy $E_T = mgr$.

At a given point on the hemisphere, the marble's potential energy is given by $mg(r - h) = mgr\cos\theta$ whilst its kinetic energy is just $mv^2/2$. This means that the marble's total energy is

$$E_T = mgr\cos\theta + \frac{1}{2}mv^2 = mgr \tag{2.42}$$

where the last equality holds due to the conservation of energy. Rearranging:

$$2mg(1 - \cos\theta) = \frac{mv^2}{r} \tag{2.43}$$

Substituting equation 2.41 into equation 2.43 gives

$$2mg - 2mg\cos\theta = mg\cos\theta - N \tag{2.44}$$

and so

$$N = 3mg\cos\theta - 2mg = mg(3\cos\theta - 2) \tag{2.45}$$

At the instant when the ball leaves the surface, there is no normal reaction force and so $N = 0$. From equation 2.45, this means that

$$3\cos\theta - 2 = 0 \tag{2.46}$$

and so finally

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) \approx 48^\circ \tag{2.47}$$

(b) Figure 2.5 illustrates the problem.

After leaving the surface of the hemisphere, the marble will now be a projectile and take a parabolic path to the ground below. We can find its velocity at the point when it leaves the surface using the conservation of energy equation 2.43:

$$2mg(1 - \cos\theta) = \frac{mv^2}{r} \to v = \sqrt{2gr(1 - \cos\theta)} \tag{2.48}$$

We know the value of $\cos\theta$ at which the marble leaves the surface from equation 2.46: $\cos\theta = 2/3$. Plugging this value in gives:

$$v = \sqrt{\frac{2gr}{3}} \tag{2.49}$$

This is the marble's initial velocity after leaving the surface of the hemisphere. It might be helpful to relabel this velocity as u since it

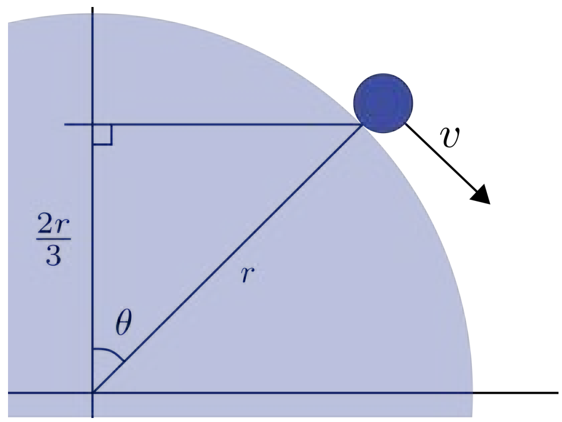


Figure 2.5: A smooth marble leaving the surface of the smooth hemisphere.

is an initial velocity, and resolving this into horizontal and vertical components gives:

$$u = \sqrt{\frac{2gr}{3}} \tag{2.50}$$

$$u_x = u\cos\theta \tag{2.51}$$

$$u_y = u\sin\theta \tag{2.52}$$

Note that for this question we will adopt the convention that *y increases in the downwards direction*. You don't have to do this, but it may make life slightly easier. (If you don't do this, then $u_y = -u\sin\theta$ and (later on) $s_y = -2r/3$.)
Referring back to the diagram, we know that the base of the triangle is $2r/3$ since $\cos\theta = 2/3$. Using Pythagoras, we can find the height x of the triangle:

$$r^2 = x^2 + \left(\frac{2r}{3}\right)^2 \rightarrow x = \frac{\sqrt{5}}{3}r \tag{2.53}$$

With this information, we can now figure out that $\sin\theta = \sqrt{5}/3$. The fact that we know $\sin\theta$ and $\cos\theta$ means we can work out explicit expressions for u_x and u_y:

$$u_x = u\cos\theta = \sqrt{\frac{2gr}{3}} \times \frac{2}{3} = \sqrt{\frac{8gr}{27}} \tag{2.54}$$

$$u_y = u\sin\theta = \sqrt{\frac{2gr}{3}} \times \frac{\sqrt{5}}{3} = \sqrt{\frac{10gr}{27}} \tag{2.55}$$

As we did in Workbook 1, it is often useful to create a table of the different variables involved in a projectile problem. Table 2.1 is an example of one way of doing this.

$s_x = ?$	$s_y = \frac{2}{3}r$
$u_x = \sqrt{\frac{8gr}{27}}$	$u_y = \sqrt{\frac{10gr}{27}}$
$v_x = \sqrt{\frac{8gr}{27}}$	$v_y = ?$
$a_x = 0$	$a_y = +g$
$t = ?$	$t = ?$

Table 2.1: A table of the different variables relevant to the *Marble rolling off a Hemisphere* problem.

$s_y = 2/3$ since this is the vertical distance the marble has to travel to hit the ground. We assume no air resistance and so there are no horizontal accelerations (hence $a_x = 0$ and $v_x = u_x$), whilst the only acceleration in the vertical direction is provided by gravity (which is *positive* due to our choice of coordinate system).

From the vertical variables in Table 2.1 we can find the time t that it takes for the marble to hit the ground using a standard SUVAT equation:

$$s_y = u_y t + \frac{1}{2}a_y t^2 \tag{2.56}$$

Substituting in the relevant values and simplifying:

$$\frac{2}{3}r = \sqrt{\frac{10gr}{27}}\,t + \frac{g}{2}t^2 \tag{2.57}$$

$$\frac{g}{2}t^2 + \sqrt{\frac{10gr}{27}}\,t - \frac{2}{3}r = 0 \tag{2.58}$$

This is a quadratic equation in t. The algebra might get a bit messy now, so make sure you are careful! Using the quadratic formula:

$$t = \frac{1}{g} \times \left(-\sqrt{\frac{10gr}{27}} \pm \sqrt{\frac{10gr}{27} + \frac{4gr}{3}} \right) \tag{2.59}$$

$$= -\sqrt{\frac{10r}{27g}} \pm \sqrt{\frac{10r}{27g} + \frac{4r}{3g}} \tag{2.60}$$

$$= -\sqrt{\frac{10r}{27g}} \pm \sqrt{\frac{46r}{27g}} \tag{2.61}$$

$$= \sqrt{\frac{2r}{27g}} \left(-\sqrt{5} \pm \sqrt{23}\right) \tag{2.62}$$

We will choose the positive root solution, as otherwise we will end up with a negative value for t. This means that

$$t = \sqrt{\frac{2r}{27g}} \left(\sqrt{23} - \sqrt{5}\right) \tag{2.63}$$

Multiplying this time by the marble's horizontal velocity u_x gives the distance it travels in the horizontal direction:

$$u_x t = \sqrt{\frac{8gr}{27}} \times \sqrt{\frac{2r}{27g}} \left(\sqrt{23} - \sqrt{5}\right) \tag{2.64}$$

$$= \sqrt{\frac{16gr^2}{27^2 \times g}} \left(\sqrt{23} - \sqrt{5}\right) \tag{2.65}$$

$$= \frac{4r}{27} \left(\sqrt{23} - \sqrt{5}\right) \tag{2.66}$$

Whilst this is the distance the marble travels horizontally, to work out how far away from the base the marble lands we need to subtract off the distance from where it leaves the sphere. Referring back to the diagram, since the height of the triangle is $\sqrt{5}r/3$ and the radius of the hemisphere is r, the marble starts a distance

$$r - \frac{\sqrt{5}}{3} r = \left(1 - \frac{\sqrt{5}}{3}\right) r \tag{2.67}$$

away from the edge of the base. So, finally, the distance d the marble lands away from the base is

$$d = \frac{4r}{27} \left(\sqrt{23} - \sqrt{5}\right) - \left(1 - \frac{\sqrt{5}}{3}\right) r \tag{2.68}$$

$$= \left(\frac{4\sqrt{23}}{27} - \frac{4\sqrt{5}}{27} + \frac{\sqrt{5}}{3} - 1\right) r \tag{2.69}$$

$$\approx 0.125r \tag{2.70}$$

8. **Problem**
   Consider a toy car going around a loop-the-loop. If the car is going too slowly around the loop-the-loop, at some point it will fall off.

   (a) If the car started at rest on a downwards ramp which was initially at the same height as the loop, would the car make it around safely? Explain why.

   (b) Bob wants to find out if it's possible to do the loop-the-loop with a real car. He has built a loop which has a 6 m radius, and his car will approach the loop driving along a flat runway. What is the minimum speed, in mph, that Bob needs to drive at in order to perform the loop-the-loop successfully?

   (c) Is there any reason why Bob shouldn't go much faster than this minimum speed?

   Hint

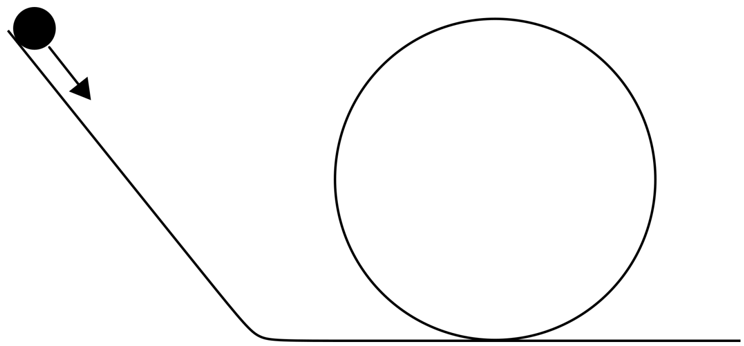


Figure 2.6: A car (black circle) at rest at the top of a downwards ramp which is the same height as the loop-the-loop.

(a) What is the velocity of the car when it gets back to its starting height? When will the car have its lowest speed during the loop? Can you find an expression for the necessary minimum speed to keep going around the loop?

(b) This question is not actually about circular motion – why do the normal equation of circular motion not apply in this scenario? Why is the speed at the bottom of the ramp different to that at the top? It may help to model the car as a marble.

(c) What are the forces a person experiences if they change direction suddenly? What could happen to their body if these forces were too large? Think about astronauts being trained in a human 'centrifuge'.

Solution

(a) No: by the conservation of energy, the car would climb back to its original height with zero kinetic energy. It would then fall straight down. In reality, it would lose contact with the track sooner than this and so would not even make it to the top of the loop.

(b) First, we need to determine the minimum velocity at the top of the loop such that the car does not fall. We can find this condition by setting the normal contact force to zero at the top of the loop. This means that:

$$mg = \frac{mv^2}{r} \tag{2.71}$$

and so

$$v_{min} = \sqrt{gr} \tag{2.72}$$

If we let the velocity of the car at the bottom of the loop be v_0, then applying the conservation of energy at the top and bottom of the loop gives:

$$\frac{1}{2}mv_{min}^2 + mgh = \frac{1}{2}mv_0^2 + 0 \tag{2.73}$$

where h is the height of the loop. This height is simply the diameter of the circle: $h = 2r$. This means that

$$v_0^2 = gr + 4gr = 5gr \tag{2.74}$$

and so

$$v_0 = \sqrt{5gr} = \sqrt{5 \times 9.81 \times 6} \approx 17\,\mathrm{ms}^{-1} \approx 38.4\,\mathrm{mph} \tag{2.75}$$

(c) Yes: the faster Bob travels, the larger the g-force he will experience as he goes around the loop. This increases his chances of passing out while the car is upside down.

9. **Problem**
A velodrome allows cyclists to travel at high speed around tight corners since the track is banked at a steep angle.

(a) By drawing a force diagram, show why this banking is necessary. Determine an expression for the maximum speed a cyclist can travel at as a function of the radius r of the corner and the coefficient of dynamic friction μ.

(b) Determine the value of μ required for a bike to travel at $80\,\mathrm{kmh}^{-1}$ around corners of radius 25 m with a maximum banking angle of 42°.

Hint

(a) Have you drawn a clear diagram showing all the forces acting on the cyclist? In which direction must the centripetal force act? Is this a sensible direction to resolve the forces?

(b) Rearrange your answer for (a) to solve for μ.

Solution

(a) Figure 2.7 shows the forces on a bicycle as it travels around a banked corner in a velodrome.
Resolving forces vertically:

$$N\cos\theta - F\sin\theta - W = 0 \tag{2.76}$$

and so, substituting $W = mg$ and $F = \mu N$,

$$N = \frac{mg}{N(\cos\theta - \mu\sin\theta)} \tag{2.77}$$

Resolving forces horizontally:

$$N\sin\theta + F\cos\theta = \frac{mv^2}{r} \tag{2.78}$$

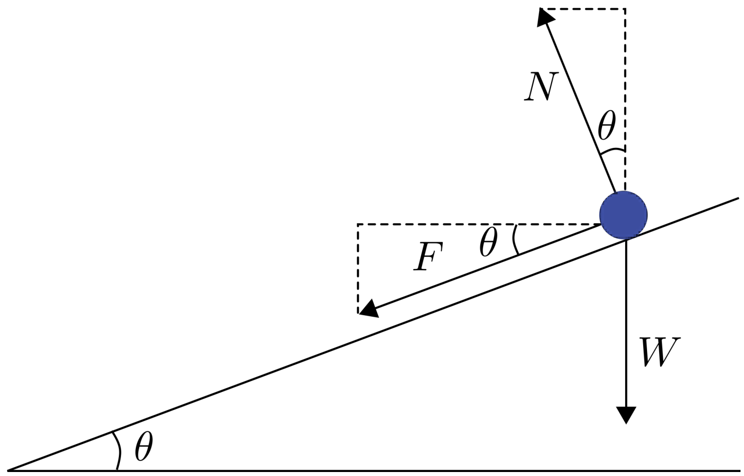


Figure 2.7: A diagram showing the forces on a bicycle as it travels around a corner banked at an angle θ.

and so, once again substituting $F = \mu N$,

$$N(\sin\theta + \mu\cos\theta) = \frac{mv^2}{r} \tag{2.79}$$

Substituting equation 2.77 into equation 2.79:

$$\frac{mg}{\cos\theta - \mu\sin\theta}(\sin\theta + \mu\cos\theta) = \frac{mv^2}{r} \tag{2.80}$$

and so, rearranging for v:

$$v_{max} = \sqrt{\frac{(\sin\theta + \mu\cos\theta)\, gr}{\cos\theta - \mu\sin\theta}} \tag{2.81}$$

(b) We want to rearrange equation 2.81 for μ:

$$v^2(\cos\theta - \mu\sin\theta) = (\sin\theta + \mu\cos\theta)\, gr \tag{2.82}$$

$$\mu\left(v^2\sin\theta + gr\cos\theta\right) = v^2\cos\theta - gr\sin\theta \tag{2.83}$$

and so

$$\mu = \frac{v^2\cos\theta - gr\sin\theta}{v^2\sin\theta + gr\cos\theta} \tag{2.84}$$

A speed of $80\,\mathrm{kmh^{-1}} \approx 22.2\,\mathrm{ms^{-1}}$. Plugging in the numbers:

$$\mu \approx \frac{22.2^2 \times \cos 42^\circ - 9.81 \times 25\sin 42^\circ}{22.2^2 \times \sin 42^\circ + 9.81 \times 25\cos 42^\circ} \approx 0.40 \tag{2.85}$$

10. **Problem**

Racing cars have spoilers which direct the air flow over the car upwards.

(a) Explain why this makes the cars 'heavier' with reference to at least one of Newton's laws of motion.

(b) A student suggests that this is a silly design feature, as the heavier an object is the slower it will go. What is the physics behind this argument, and is it correct?

(c) Assume that the downforce created from the spoilers is proportional to v^2, where v is the velocity of the car. If it is possible for a racing car of mass 800 kg to drive on the roof of a tunnel provided it is travelling faster than 150 mph, determine the value of the constant of proportionality.

Hint

(a) Newton's third law is important here.

(b) Although there is inertial mass and gravitational mass (which just so happen to be the same thing), does the weight of the car actually increase?

(c) What is the relationship between the weight of the car and the 'downforce'?

Solution

(a) By Newton's third law, if the air is pushed upwards off the spoiler, the spoiler must likewise be pushed down by the air. This extra downwards force (or *downforce*) adds to the weight of the car and pushes it down. This in turn increases both the normal reaction force and the frictional forces.

(b) The argument is not correct. Whilst the spoiler affects the downwards force (and therefore the weight) of the car, the inertial mass of the car (that is, the mass relevant to $F = ma$) does not change. This means that it takes the same amount of force to accelerate the car with or without a spoiler.

Since the spoiler increases the weight of the car, the normal reaction force N is also increased. The frictional force F between the tyres and the road is given by the standard equation

$$F = \mu N \tag{2.86}$$

where μ is the coefficient of dynamic friction. This in turn means that the spoilers increase the friction between the tyres and the road. The more friction there is between the tyres and the road, the more grip the car will have. (This is why racing cars often have dry 'slick' tyres with no grooves on them – to improve grip by maximising the amount of contact the tyre has with the road.) This increased grip will in turn improve the acceleration of the car.

(c) $150\,\text{mph} \approx 67\,\text{ms}^{-1}$. Letting the constant of proportionality be k, balancing the gravitational force with the downforce gives:

$$mg = kv^2 \tag{2.87}$$

and so

$$k = \frac{mg}{v^2} \approx \frac{800 \times 9.81}{67^2} \approx 1.75 \tag{2.88}$$

11. **Problem**

In Hertford, Hampshire and Hereford, hurricanes hardly happen. However, if they were to happen, would they likely rotate clockwise, anti-clockwise or have no preference?

Hint

You should piece together the physics behind this phenomenon. Think about:

- Most weather patterns are driven by the Sun.
- What is the significance of isobars on a weather forecast?
- Where do hurricanes usually occur and why?
- What do the above questions have to do with circular motion?
- What is the Coriolis effect?

Solution

This question is about the Coriolis effect: when air travels around the Earth, it is subject to the *Coriolis force*, an apparent force (like the centrifugal force) which appears due to the rotation of the Earth.

The effect of the Coriolis force can be illustrated by visualising throwing a paper aeroplane from the equator northwards. We will neglect any wind and any air resistance, and just focus on the effect of the Earth's rotation.

A person standing on the equator is rotating faster around the Earth's core than someone standing at the poles, simply due to the geometry of the Earth. Likewise, the paper aeroplane which starts at rest at the equator is rotating around the Earth's core faster than if it were more north. Another way of expressing this idea is to say that the paper aeroplane has more angular momentum than if it were to start more northwards. This angular momentum which the paper aeroplane has must be conserved.

As the paper aeroplane travels northwards, it flies over land which is rotating slower than the equator. Since the aeroplane conserves its angular momentum, it effectively keeps its rotational velocity around the Earth's core that it had at the equator. The effect of this is that the paper aeroplane starts to appear to bend rightwards as it travels over the Earth's surface.

A similar thought experiment going the other way will show that a paper aeroplane which starts off at the North pole and travels southwards will also appear to bend rightwards (in its direction of travel) as it travels over the Earth's surface. In the Southern hemisphere, the paper aeroplane would instead bend leftwards.

The heating of the oceans and land by the Sun causes the air to warm, which then circulates following areas of higher and lower pressure.

Imagine an area of low pressure which develops a little north of the equator. Higher pressure air will then move towards this patch of low pressure from all directions.

Picturing this with the knowledge that air always bends rightwards as it travels in the Northern hemisphere due to the Coriolis effect, we can see that the air starts circulating clockwise in the Northern hemisphere. On the other hand, air starts circulating anticlockwise in the Southern hemisphere.

This means that if hurricanes were to happen in Hertford, Hampshire and Hereford, they would likely rotate clockwise.

12. **Problem**

For an object undergoing simple harmonic motion, it is possible to express its velocity as a function of time (as it is simply the time derivative of its displacement). Show that an objecting undergoing simple harmonic motion has a velocity as a function of displacement is given by

$$v(x) = \pm\omega\sqrt{A^2 - x^2} \tag{2.89}$$

where all the symbols have their usual meanings.

Hint

You should be able to derive the fact that $v(t) = -A\omega\sin(\omega t)$. There is a useful trigonometric identity which will help to get rid of the sin and cos terms, if you square them.

Solution

The displacement of an object undergoing simple harmonic motion is given by

$$x = A\cos(\omega t) \tag{2.90}$$

where A is the amplitude, ω is the angular frequency and t is time. Differentiating with respect to time:

$$v = -A\omega\sin(\omega t) \tag{2.91}$$

Squaring both the displacement and the velocity gives $x^2 = A^2\cos^2(\omega t)$ and $v^2 = A^2\omega^2\sin^2(\omega t)$. Recognising the fact that we would like to use the trigonometric identity $cos^2x + \sin^2 x = 1$:

$$x^2 + \frac{v^2}{\omega^2} = A^2\left[\cos^2(\omega t) + \sin^2(\omega t)\right] = A^2 \tag{2.92}$$

Rearranging this gives the final result:

$$v(x) = \pm\omega\sqrt{A^2 - x^2} \tag{2.93}$$

Note that you could equivalently start from the fact that displacement is given by $x = A\sin(\omega t)$ and proceed with the problem in an identical way to get the same solution.

13. **Problem**

    Imagine that a tunnel is constructed straight through the centre of the Earth. If a person were to fall into the tunnel, would they arrive at the other end? Describe the motion of the person and *either* explain why the person would not reach the other end *or* calculate the time taken for the person to travel from one end of the Earth to the other.

    Hint

    What happens to the force due to gravity as the person approaches the centre of the Earth? As $r \to 0$, does the force not become infinite? What effect does this have on their velocity? How fast will they be travelling when they get to the centre of the Earth?

    It turns out that we can ignore all the mass at a greater radius from the centre of the Earth than the person is at any given point. This is because the gravitational pull from all of the material contained within this ring exactly cancels out. As the person falls towards the centre of the Earth, there is some mass above them, which is now pulling them upwards. But this exactly cancels out the pull of the other mass outside radius r.

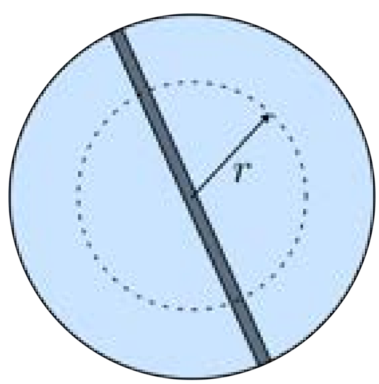


Figure 2.8: If the person is at radius r, then the gravitational pull from all the mass of the Earth contained outside of this radius exactly cancels out.

In other words, it is only the mass that is contained within a sphere of radius equal to the person's displacement from the centre of mass that contributes to the force of gravity. This means that you only have to consider the mass within the person's radius.

Recall that simple harmonic motion occurs if the acceleration of an object is proportional to, and in the opposite direction to, the object's displacement about the equilibrium position.

Solution

The person would arrive at the other end – this is a classic example of simple harmonic motion.

It is a fact that it is only the mass that is contained within a sphere of radius equal to the person's displacement from the centre of mass that contributes to the force of gravity.

This leaves a linearly decreasing gravitational field strength, whereby the gravitational force the person is subject to at any one time is:

$$F = \frac{m_E(r)m_p}{r^2} \tag{2.94}$$

where m_p is the mass of the person and

$$m_E(r) = \frac{4}{3}\pi r^3 \rho \tag{2.95}$$

is the mass contained within the spher of radius equal to the person's displacement from the centre of the Earth.

Substituting equation 2.95 into 2.94 gives a force which is linearly dependent on radius:

$$F(r) = \frac{4\pi G m_p \rho}{3} r \tag{2.96}$$

The explicit expression for the density of the Earth is

$$\rho = \frac{\text{mass}}{\text{volume}} = \frac{3M_E}{4\pi R_E^3} \tag{2.97}$$

where M_E and R_E are the total mass and radius of the Earth. Inputting this into equation 2.96 gives:

$$F(r) = \frac{GM_E m_p r}{R_E^3} \tag{2.98}$$

This is the force on the person at any given point, which is also equal to

$$F(r) = m_p a \tag{2.99}$$

by Newton's second law. Equating equations 2.98 and 2.99 and solving for the acceleration:

$$a = -\frac{GM_E}{R_E^3} r \tag{2.100}$$

Since this acceleration is proportional to the person's displacement r and in the opposite direction to it, the person is subject to simple harmonic motion. The defining equation of simple harmonic motion

$$a = -\omega^2 x \tag{2.101}$$

leads to an angular frequency of

$$\omega = \sqrt{\frac{GM_E}{R_E^3}} \tag{2.102}$$

Since

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{2.103}$$

this leads to a time period of

$$T = 2\pi\sqrt{\frac{R_E^3}{GM_E}} \tag{2.104}$$

Plugging in standard values for the constants:

$$T = 2\pi\sqrt{\frac{(6.4\times10^6)^3}{6.7\times10^{-11}\times6\times10^{24}}} \approx 5100\,\text{seconds} \approx 85\,\text{minutes} \tag{2.105}$$

Note that this is the time it would take to start from one end, fall to the other end and then return to your original starting point. This means that the time it would take for the person to travel from one end of the Earth to the other is half of this, or roughly 42 minutes.

14. **Problem**
In The A-Team film, Hannibal and his team find themselves plummeting towards the Earth in a tank with only one of its three parachutes attached. This would not be a soft landing! However, there is a lake about half a mile away from their landing spot. The team attempt to 'fly the tank' to the lake by firing shells horizontally. This question will examine whether this is pure Hollywood or based in sound physics.

For the team to be successful, how high up must they be when the execute this plan? You may ignore the effects of air resistance in the horizontal direction.

The following data may be useful:

- Projectile mass: 10 kg
- Muzzle velocity: 1750 ms^{-1}
- Time between shots: 3.5 s
- Tank mass: 22 000 kg
- Terminal velocity: 33 mph

Hint
Make as many simplifying assumptions as you can. How important is it that the mass of the tank will decrease?

Can you smooth out the force? Instead of having many impulses every 3.5 s, consider finding an average continuous force.

If you can find a constant acceleration then regular SUVAT equations can be applied.

Solution
Since force is the rate of change of momentum, the average continuous force provided to the tank by firing the shells is

$$F = \frac{\Delta p}{\Delta t} = \frac{1750 \times 10}{3.5} = 5000\,\text{N} \tag{2.106}$$

By Newton's second law, this provides an average (constant) acceleration of

$$a = \frac{F}{m} = \frac{5000}{20000} = 0.25\,\text{ms}^{-2} \tag{2.107}$$

where we have assumed that m, the mass of the tank, is constant given the fact that the mass of the projectiles being fired is much less than the mass of the tank.

Now we can use a standard SUVAT equation

$$s = ut + \frac{1}{2}at^2 \tag{2.108}$$

in the horizontal direction to work out the time of flight. Assuming that the initial horizontal velocity of the tank is $u = 0\,\text{ms}^{-1}$ and plugging in a horizontal distance of $s \approx 800\,\text{m}$, the time taken by the tank to fly to the lake is

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 800}{0.25}} = \sqrt{6400} = 80\,\text{s} \tag{2.109}$$

Now all that is left to do is figure out how far vertically the tank falls in this time. This will be the height the team must be when they execute this plan in order to get to the lake before crashing into the ground.

Using the fact that $33\,\text{mph} \approx 14.75\,\text{ms}^{-1}$:

$$\text{distance} = \text{speed} \times \text{time} = 14.75 \times 80 \approx 1200\,\text{m} \tag{2.110}$$

It is worth reflecting on our assumptions: the tank will inevitably lose mass due to the $80/3.5 \approx 23$ projectiles fired. We have also neglected the effect of sideways air resistance and any wind. Finally, we assumed that the tank's initial horizontal velocity was zero; this may not be the case due to the initial horizontal velocity from the plane.

Chapter 3

Waves and Optics

3.1 Introductory Problems

1. **Problem**
   A triangular glass prism sits on a table pointing upwards. A beam of coloured light is directed horizontally near the top of the prism, as shown in Figure 3.1. What happens to the light beam at the prism?

   (a) It is bent upwards
   (b) It is bent downwards
   (c) It continues horizontally
   (d) It depends on the colour of the light

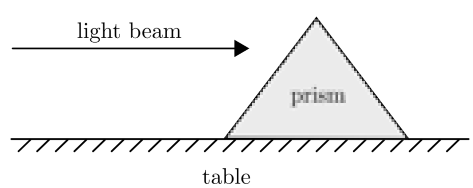


Figure 3.1: A beam of coloured light directed horizontally towards the top of a triangular glass prism.

   Hint
   Read the question carefully – it is a beam of coloured light, for example red light or green light. Remember the dispersion of white light by a prism. What happens to the white light and why? Does it disperse by a lot or only by a little? Draw a ray diagram.

   Solution
   No matter the colour, any light beam will follow the normal rules of refraction and bend towards the normal when entering a more dense medium. This means that the correct answer is (b): it is bent downwards.

2. **Problem**
A beam of light is incident from a vacuum onto a medium at an angle θ to the normal of the boundary. The refracted and partially reflected beams happen to form a right angle. Find an expression for the refractive index of the medium.

Hint
Draw a digram showing reflection and refraction at the surface. Label angles (both known and unknown). Use Snell's law and remember that $\sin(90^\circ - \theta) = \cos\theta$.

Solution
See Figure 3.2.

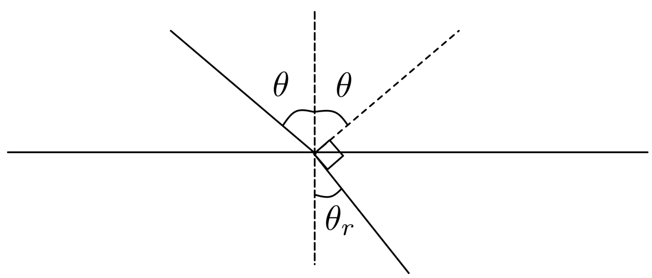


Figure 3.2: A beam of light incident from a vacuum onto a medium at an angle θ to the normal of the boundary.

If the angle of refraction is θ_r, then

$$\theta + \theta_r = 90^\circ \tag{3.1}$$

and so

$$\theta_r = 90^\circ - \theta \tag{3.2}$$

Let the refractive index of the medium be n. Since the refractive index of a vacuum is 1, using Snell's law:

$$n \sin\theta_r = \sin\theta \tag{3.3}$$

If we rearrange this equation for n and substitute for θ_r using equation 3.2, this gives our final answer:

$$n = \frac{\sin\theta}{\sin\theta_r} = \frac{\sin\theta}{\sin(90^\circ - \theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta \tag{3.4}$$

3.2 Further Problems

3. **Problem**
This question concerns total internal reflection, optical fibres, and refraction. You may assume that the refractive index of glass is larger than that of water, and that the refractive index of water is larger than that of air.

(a) Explain what is meant by the phrases *total internal reflection* and *critical angle*. (You are encouraged to use a diagram to explain your answer.)

(b) Derive an equation relating the critical angle and the refractive indices of two materials, n_1 and n_2, where $n_2 < n_1$.

(c) An optical fibre is usually made of two materials, a core and a cladding, as shown in Figure 3.3 (not drawn to scale).

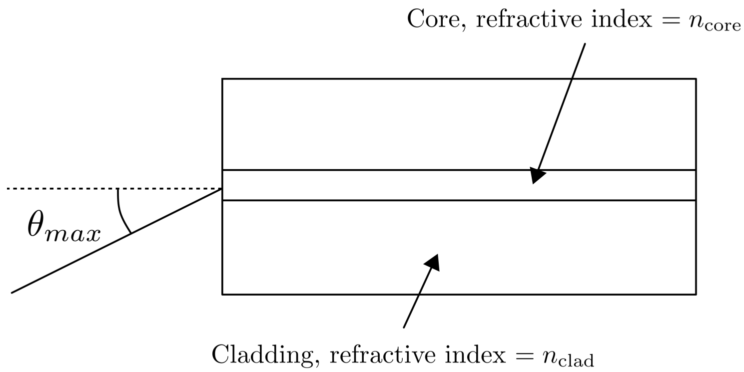


Figure 3.3: A diagram of an optical fibre.

Light may only be transmitted along the fibre if the incident angle of the light is less than a maximum angle θ_{max}. By using your equation from above and Snell's Law, or otherwise, derive an expression for θ_{max} in terms of the core and cladding refractive indices only.

Hint

(a) Draw a clear diagram to illustrate total internal reflection and critical angle with labels.

(b) For the derivation, start by using Snell's law. What is the angle of refraction for light if the angle of incidence is equal to the critical angle? Optical fibres contain a core and cladding where $n_{core} > n_{cladding}$. This allows total internal reflection.

(c) Redraw the ray diagram for light entering the cone from air at θ_{max}. Show the refraction the light undergoes as it enters the core, and then as it hits the cladding at the critical angle (the light is *just* transmitted at this point). Label angles of incidence, reflection and refraction using the standard formulae and trigonometry.

Use your derived critical angle formula at the point where the light hits the cladding, and use Snell's law where the ray enters the core with the relevant refractive indices. Also remember standard trigonometric rules and substitutions.

Solution

(a) See Figure 3.4.

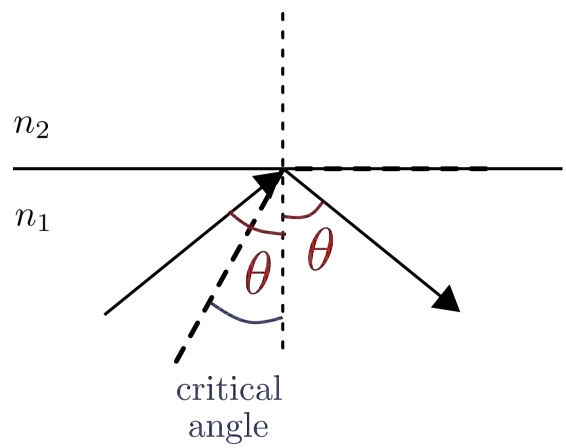


Figure 3.4: A diagram illustrating the concepts of total internal reflection and critical angle.

Imagine a light ray travelling in a more dense medium approaching a boundary with a less dense medium. Depending on the angle of incidence and the respective refractive indices, some of the light may be reflected back into the more dense medium and some of the light may be refracted into the less dense medium.

Total internal reflection is the complete reflection of light within a medium, and occurs if the angle of incidence is greater than the critical angle.

The *critical angle* is the minimum angle of incidence at which rays suffer total internal reflection.

(b) First consider Snell's law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{3.5}$$

At the critical angle θ_c, this becomes:

$$n_1 \sin\theta_c = n_2 \sin 90^\circ \tag{3.6}$$

and so

$$\sin\theta_c = \frac{n_2}{n_1} \tag{3.7}$$

(c) Consider Figure 3.5.

At point A:

$$\sin\theta_c = \frac{n_{clad}}{n_{core}} \tag{3.8}$$

At point B, using Snell's law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{3.9}$$

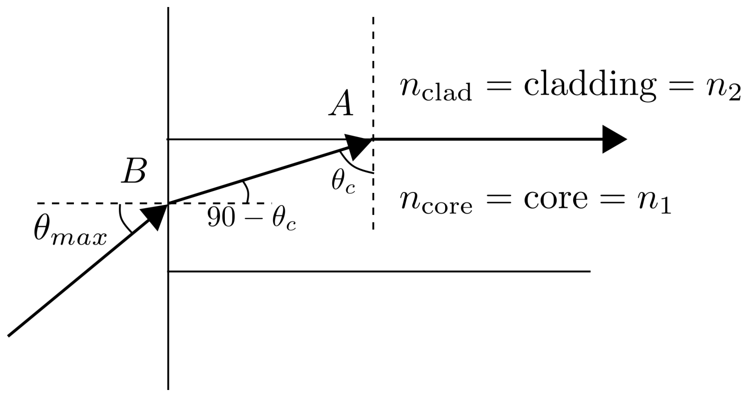


Figure 3.5: A diagram showing the passage of light through the optical fibre.

$\theta_1 = \theta_{max}$ and $\theta_2 = 90° - \theta_c$. Approximating the refractive index of air as $n_1 = 1$ and letting $n_2 = n_{core}$:

$$\sin\theta_{max} = n_{core}\sin(90° - \theta_c) \tag{3.10}$$

$$= n_{core}\cos\theta_c \tag{3.11}$$

$$= n_{core}\sqrt{1 - \sin^2\theta_c} \tag{3.12}$$

Substituting for $\sin\theta_c$ using equation 3.8:

$$\sin\theta_{max} = n_{core}\sqrt{1 - \left(\frac{n_{clad}}{n_{core}}\right)^2} = \sqrt{n_{core}^2 - n_{clad}^2} \tag{3.13}$$

and so our final answer is:

$$\theta_{max} = \sin^{-1}\left(\sqrt{n_{core}^2 - n_{clad}^2}\right) \tag{3.14}$$

4. **Problem**
In an optical fibre, light can travel directly down the middle of the fibre. Alternatively, a *meridional ray* is one which bounces off the walls of the fibre yet stays in a single plane. The minimum angle a ray can bounce at is controlled by the critical angle. For a glass fibre with a core index of 1.500, a cladding index of 1.496 and length 1 km:

   (a) Calculate the maximum path length for the meridional ray.

   (b) Hence calculate the time difference for this ray and a ray which passes straight through.

   (c) If square (in time) pulses of light are used to send information down the fibre, calculate the maximum rate at which information can be sent.

Hint

(a) Use refractive indices to calculate the critical angle for the optical fibre, remembering that $n_{core} > n_{cladding}$ and substituting as appropriate.

Do you need to be concerned about the multiple reflections to calculate the maximum path length, or can you make a straightforward assumption?

The maximum path length will occur when the angle of incidence is equal to the critical angle at the cladding, at the far end of the optical fibre. Use a triangle and trigonometry to find the maximum path length.

(b) When calculating the time difference, remember that you will need to use the speed of light *in the medium*, which you can determine using the refractive index of the medium.

(c) Draw a square pulse. What assumption can you make in terms of the size of the gap needed to avoid interference but send information at a maximum rate? Use this to calculate the number of pulses per second (which is the maximum rate at which information is sent).

Solution

(a) The maximum path length occurs if the ray hits the cladding at the critical angle. From a previous question we know that the critical angle is given by

$$\sin\theta_c = \frac{n_{clad}}{n_{core}} = \frac{1.496}{1.500} \tag{3.15}$$

and so $\theta_c = 85.81°$.

Consider the triangle shown in Figure 3.6.

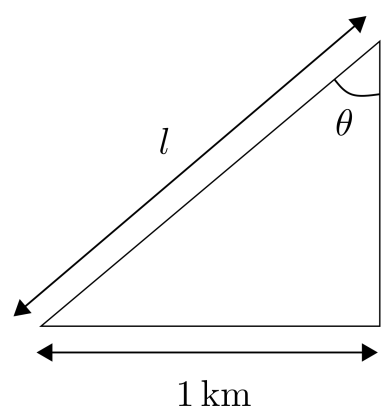


Figure 3.6: A helpful triangle. Angle $\theta = 85.81°$ and length l corresponds to the maximum path length.

The maximum path length is given by the hypotenuse of the triangle, and so

$$\text{maximum path length} = \frac{1}{\sin 85.81^\circ} = \frac{1.500}{1.496} = 1.002\,67\,\text{km} \quad (3.16)$$

(b) The path difference Δd between two rays is

$$\Delta d = 1.00267 - 1 = 0.002\,67\,\text{km} = 2.67\,\text{m} \quad (3.17)$$

The time difference is then given by $\Delta t = \Delta d/c_s$, where c_s is the speed of the ray in the medium. Since the refractive index n of a medium and the speed of light c_s within the same medium are related by

$$n = \frac{c}{c_s} \quad (3.18)$$

where c is the speed of light in a vacuum, this means that

$$c_s = \frac{c}{1.500} \quad (3.19)$$

Plugging in $c \approx 3 \times 10^8$ ms^{-1} then gives

$$\Delta t = \frac{2.67}{c_s} \approx 2.67 \times \frac{1.500}{3 \times 10^8} \approx 1.335 \times 10^{-8}\ \text{s} \quad (3.20)$$

(c) A square pulse is depicted in Figure 3.7.

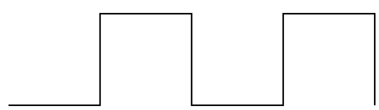


Figure 3.7: A square pulse.

We can assume a *minimum* gap of $\Delta t = 1.335 \times 10^{-8}$ s between pulses, regardless of the peak length. For there to be no overlap, the maximum rate is simply the number of pulses per second:

$$\text{maximum rate} = \frac{1}{1.335 \times 10^{-8}} = 7.49 \times 10^7\ \text{s}^{-1} \quad (3.21)$$

or around 75 million pulses per second.

5. **Problem**
In a particle physics experiment, light from a particle detector is to be collected and concentrated by reflecting it between a pair of plane mirrors with angle 2α between them, as shown in Figure 3.8. A faint parallel beam of light consisting of rays parallel to the central axis is to be narrowed down by reflection off the mirrors, as shown by the single ray illustrated, for which angle $a = \alpha$.

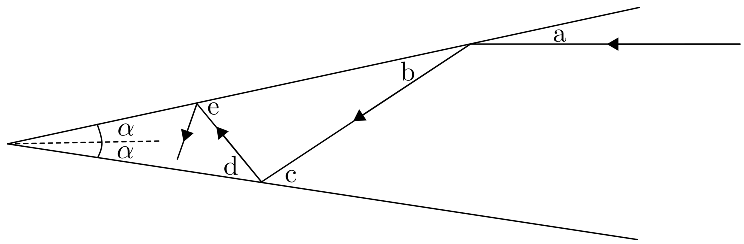


Figure 3.8: A parallel beam of light being reflected between a pair of plane mirrors.

(a) Determine angles b, c, d and e in terms of angle α.

(b) Explain what happens after several reflections of the light down the mirror funnel.

(c) If $\alpha = 10°$, what is the total number of reflections between the mirrors that will be made by a beam of light entering parallel to the axis of symmetry as shown?

(d) If the mirrors are replaced by an internally silvered circular cone whose cross-section is the same as that shown above, why will this not make any difference to the calculation given above for the plane angled mirrors with a beam of light parallel to the axis?

(e) An ear trumpet was a device that was used to collect sound and focus it into the ear. It was a cone about 0.5 m long with an angle 2α of about 30°. The sound passing into the device would typically have a frequency of 400 Hz and a speed of $330\,\mathrm{m\,s^{-1}}$. Why is the model above that we have used for light not valid for an ear trumpet used to collect sound?

Hint

(a) Draw your own ray diagram. Angle a is given. Angle b should be straightforward to figure out using the law of reflection. Angles c and d are also reflections but are as yet unknown. Use standard trigonometry rules to do with angles on a straight line and angles in a triangle. You will end up with a set of simultaneous equations: see what you can eliminate and solve for c, d and e.

(b) Consider the pattern in the angles which you have just calculated. It may help to redraw the diagram, adding more reflections.

(c) Once again, use the pattern in the angles.

(d) Draw a cone. Does anything change?

(e) Calculate the wavelength of the sound waves. How does this compare to the size of the cone? What does this tell you?

Solution

(a) Consider Figure 3.9.

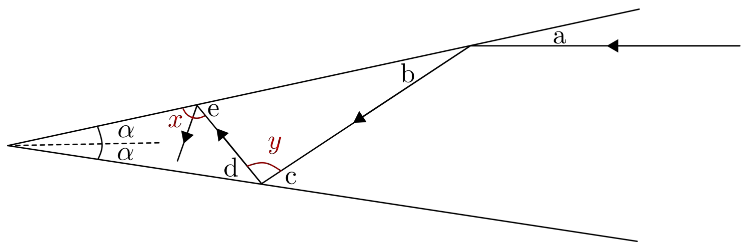


Figure 3.9: A parallel beam of light being reflected between a pair of plane mirrors with additional angles labelled.

Since the angle of incidence is equal to the angle of reflection, $b = a = \alpha$.

For the same reason, $c = d$. Let the angle between angles c and d be angle x. Then

$$x + e + b = 180^\circ \tag{3.22}$$

due to angles in a triangle, and

$$c + x + d = 180^\circ \tag{3.23}$$

due to angles on a straight line. Finally, let the angle which forms a straight line with angle e be angle y. Then

$$y + e = 180^\circ \tag{3.24}$$

due to angles on a straight line, and

$$2\alpha + d + y = 180^\circ \tag{3.25}$$

due to angles in a triangle.

Equations 3.22–3.25 form a set of simultaneous equations. Although there are four equations but six unknowns, the facts that $c = d$ and $b = \alpha$ brings the number of unknowns down to four (not counting α as an unknown, since the final answers will have to be given in terms of α). These can be solved in the usual way to get:

$$b = \alpha \tag{3.26}$$

$$c = 3\alpha \tag{3.27}$$

$$d = 3\alpha \tag{3.28}$$

$$e = 5\alpha \tag{3.29}$$

(b) After several reflections of the light down the mirror funnel, the angle of incidence becomes greater than 90° and so the ray reflects back out of the opening.

(c) The reflections off the sides follow the pattern $a = \alpha$, $c = 3\alpha$, $e = 5\alpha$ and so on. So in this case, $a = 10°$, $c = 30°$, $e = 50°$, and then the next two reflections will be of angles $70°$ and $90°$.

After this point, the ray will reflect back (following the same path for this particular angle). This results in a total of $4 + 1 + 4 = 9$ reflections.

(d) This would not make a difference since the incident ray, the normal and the reflected ray all lie in a plane containing the axis of symmetry.

(e) The sound waves would have a wavelength of

$$\lambda = \frac{v}{f} = \frac{330}{400} \approx 0.8\,\mathrm{m} \tag{3.30}$$

Since the wavelength of the sound waves is similar to the size of the trumpet aperture, diffraction will be important and needs to be taken into account.

3.3 Extension Problems

6. **Problem**
   Consider the diagram in Figure 3.10. Indicate clearly the position and nature of the image formed by the mirror. Draw rays corresponding to light coming from the open circle, and mark any relevant angles.

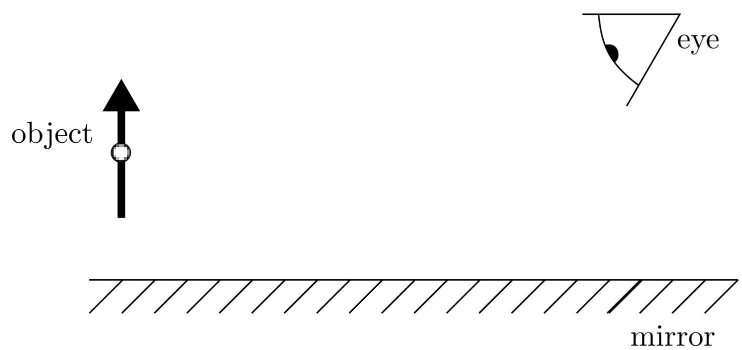


Figure 3.10: Indicate clearly the position and nature of the image formed by the mirror. Draw rays corresponding to light coming from the open circle, and mark any relevant angles.

   Hint
   Use a ruler to draw a ray diagram – what type of image is formed in a mirror?

   Solution
   The solution is presented in Figure 3.11. Angles i and r denote the angles of incidence and reflection respectively.

7. **Problem**
   A parallel sided slab of medium B and refractive index n_B is sandwiched between two slabs of medium A of refractive index n_A. A beam of light passes from A through B and into A on the other side. If the beam is

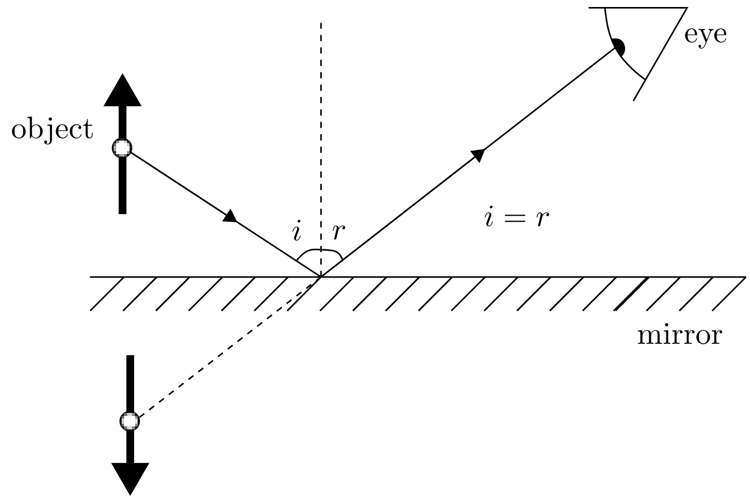


Figure 3.11: The solution to the task in Figure 3.10. Angle i is the angle of incidence and angle r is the angle of reflection.

incident on B at an angle of θ to the normal, what is the angle to the normal of the light beam in A after it has left B?

(a) $\cos^{-1}\left(\frac{n_A \sin\theta}{n_B}\right)$

(b) θ

(c) $\sin^{-1}\left(\frac{n_A^2 \sin\theta}{n_B^2}\right)$

(d) $\sin^{-1}\left(\frac{n_A \sin\theta}{n_B}\right)$

(e) $\frac{n_A}{n_B}\theta$

Hint
Find the refraction angle as the light beam leaves the second surface (going from B to A). Draw a ray diagram and label refractive indices and angles. Use Snell's law and the alternate angle rule.

Solution
See Figure 3.12.

Snell's law gives us two equations:

$$n_A \sin\theta = n_B \sin\theta_1 \tag{3.31}$$

$$n_B \sin\theta_2 = n_A \sin\theta_3 \tag{3.32}$$

Since θ_1 and θ_2 are alternate angles, $\theta_1 = \theta_2$. This means that

$$n_A \sin\theta = n_A \sin\theta_3 \tag{3.33}$$

and so $\theta_3 = \theta$. So the correct answer is (b).

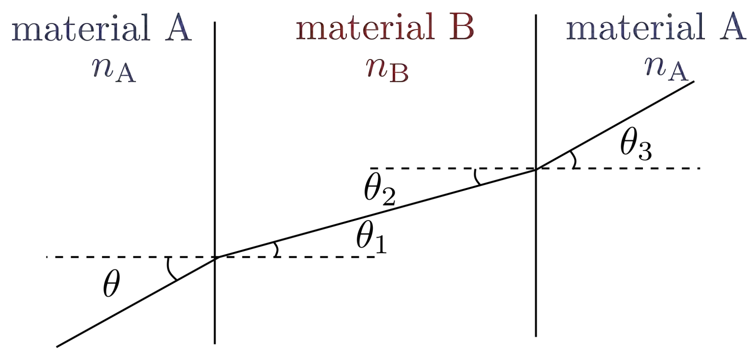


Figure 3.12: A beam of light passing through multiple media.

8. **Problem**
   A parallel beam of monochromatic light, initially travelling in a direction above the horizontal, enters a region of atmosphere in which the refractive index increases steadily with height. Which of the graphs in Figure 3.13 represents the path of the beam of light?

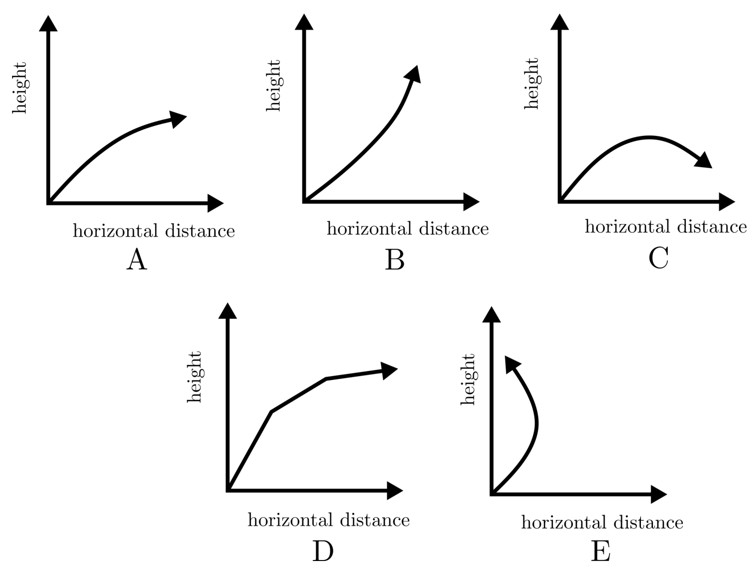


Figure 3.13: Which of these graphs represents the path of the beam of light?

   Hint
   Make sure you pay attention to the fact that the refractive index increases with height. Think about the atmosphere consisting of many thin layers of different refractive indices, with each layer having a refractive index slightly greater than the one below. Draw a diagram showing refraction

through one layer, and then consider multiple layers.

Solution
Treat the atmosphere as a set of small slabs. Since the refractive index increases with height, as the light travels up through one slab that its path will look roughly like Figure 3.14.

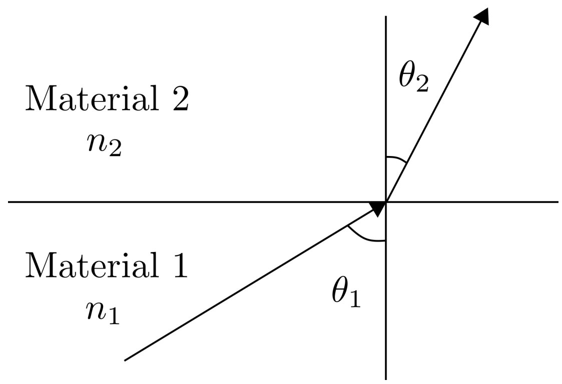


Figure 3.14: Light refracting as it passes from one slab to another.

From Snell's law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{3.34}$$

and so since $n_2 > n_1$, $\sin\theta_1 > \sin\theta_2$. This in turn means that $\theta_1 > \theta_2$ and so the ray bends upwards. In the limit where the small slabs have negligible thickness, this bending towards the normal becomes a smooth curve. This means that graph B is correct.

9. **Problem**
A narrow beam of light is incident normally upon a thin slit. The light that passes through is spread out by diffraction. The thin slit is then immersed in a container of water. The beam of light is shone through the water and is again at normal incidence to the slit. The spread of the diffracted beam of light in water will be:

   (a) The same as in air
   (b) Diffraction will not occur in water
   (c) Less spread out than in air
   (d) More spread out than in air

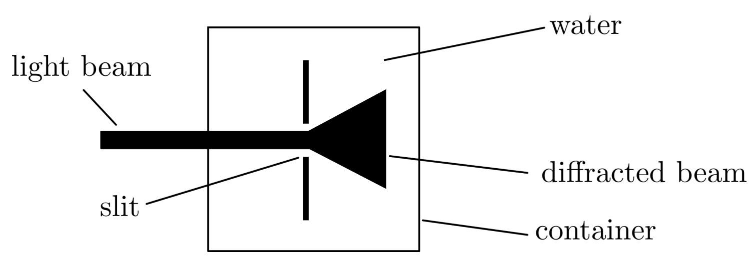


Figure 3.15: A narrow beam of light incident normally upon a thin slit in water.

Hint
Consider how the refractive index of water changes the refraction – and therefore diffraction – of light through the slit.

Solution
The refractive index of water is higher than that of air. When light passes between media, its frequency stays the same. Since

$$\text{wave speed} = \text{frequency} \times \text{wavelength} \tag{3.35}$$

and light's speed decreases when entering a more dense medium such as water, the wavelength of light will be higher in water than in air.

Diffraction is more effective on shorter wavelengths of light than longer wavelengths. Hence the spread of the diffracted beam in water will be less spread out than in air. The correct answer is (c).

10. **Problem**
Figure 3.16 shows two mirrors X and Y, and a solid object with white spots at P and Q.

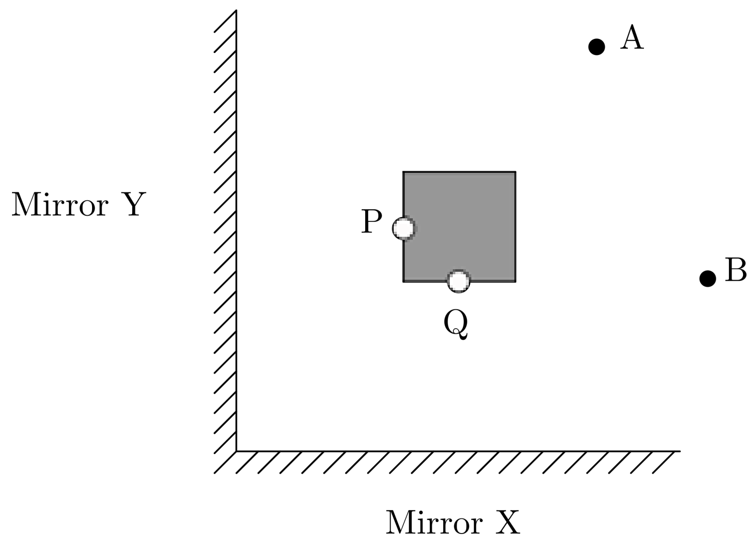


Figure 3.16: Two mirrors X and Y and a solid object with white spots at P and Q.

(a) An observer at A sees an image of P reflected in mirror Y. Mark R, the position of this image, and draw a ray from P to the observer at A.

(b) In which mirror would an observer at A see an image of spot Q? Mark S, the position of this image.

(c) An observer at B can see an image of P resulting from reflections at *both* mirrors. Draw a ray of light from P to B which enables this image to be seen.

Hint
Consider where reflection occurs when a ray travels between two points, and where the observer sees the image. Label the diagram carefully.

Solution

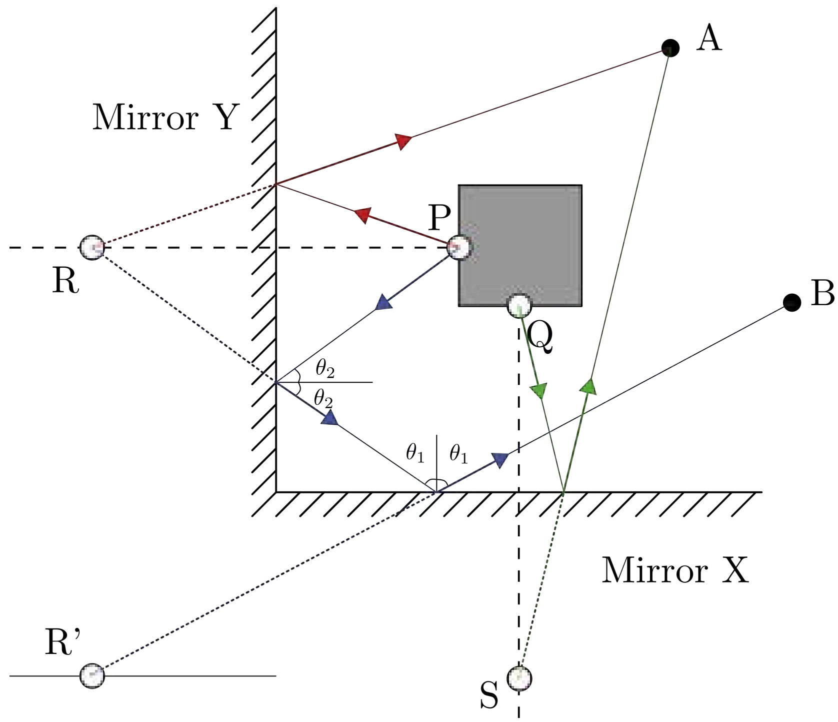


Figure 3.17: The same diagram as in Figure 3.16 but labelled to illustrate the answers to the question.

11. **Problem**
    A fisherman listens to the radio as he sits on the bank waiting for a fish to bite. The sound is also heard by the fish and the path of the sound waves entering the water is shown in Figure 3.18.

    (a) Describe what happens to the frequency, wavelength and speed of sound as it moves from air to water.

    (b) The fisherman's radio has two speakers, as shown in Figure 3.19.

    Sketch a diagram illustrating how destructive interference between sounds from the two speakers can occur when the radio is playing a note of a single frequency, assuming that the waves from the two speakers start in phase.

    Hint

    (a) Which property of waves remains invariant as waves travel between media? What does the path of the sound waves tell you about the waves' speed?

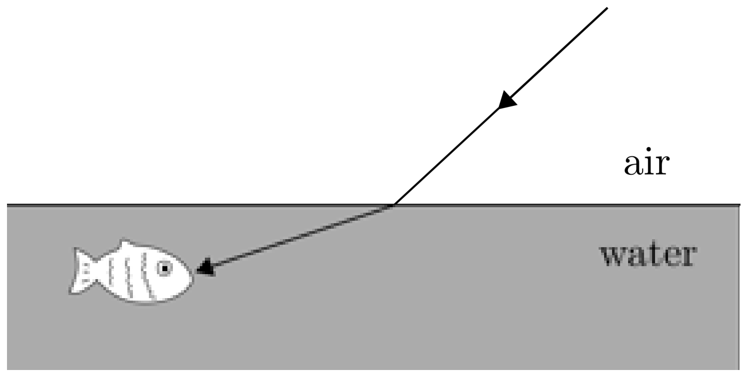


Figure 3.18: The path of the sound waves entering the water.

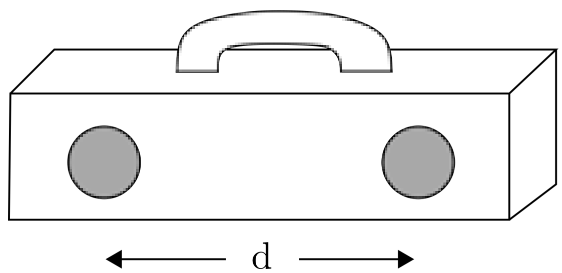


Figure 3.19: The speaker of the fisherman's radio.

(b) If the waves start in phase, how they can end in antiphase and thus result in destructive interference?

Solution

(a) As sound moves from air to water, its frequency remains the same. What happens to its speed can be determined in at least two ways:

- In Figure 3.18, the sound waves bend away from the normal. This means that the speed of sound increases in water.
- Sound waves travel through a medium via the vibrations of molecules. Since the molecules in water are closer together than the molecules in air, sound must travel faster in water than in air. By a similar argument, sound waves travel even faster through solids.

Finally, since

$$\text{wave speed} = \text{frequency} \times \text{wavelength} \tag{3.36}$$

the wave's wavelength also increases.

(b) See Figure 3.20.

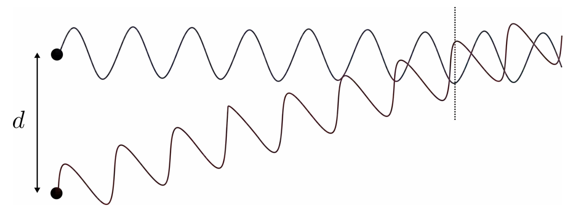


Figure 3.20: A diagram illustrating how destructive interference can occur between sounds from the two speakers.

12. **Problem**

(a) Intensity decays as one moves further away from a source, due to the rays diverging. If I is the intensity and r is the distance from the source, then $I \propto r^n$ for what value of n?

(b) Rayleigh scattering is an effect that causes many optical phenomena. It is caused by the scattering of light by small particles, such as molecules that make up the air in the atmosphere.

If a beam of intensity I_0 and wavelength λ interacts with one of these particles, then the intensity of the light scattered at an angle θ is proportional to

$$I_0 \lambda^m r^n \alpha^6 (1 + \cos^2 \theta) \tag{3.37}$$

where r is the distance from the scattering particle and α is the diameter of the scattering particle. The relationship between the intensity of the scattered light (for a given wavelength) with the distance from the scattering particle is the same as for a point source. By considering the dimensions of the quantities involved, what is m to one significant figure?

Hint

(a) Does intensity increase or decrease with distance? Can you remember what kind of law it follows? Other examples of this kind of law include Newton's law of gravitation and Coulomb's law.

(b) Find the dependence of intensity on wavelength using dimensional analysis by using n from part (a). What are the units? Does it matter if you don't know the units for intensity?

Solution

(a) Intensity follows an inverse square law, so

$$I \propto \frac{1}{r^2} \tag{3.38}$$

and $n = -2$.

(b) Letting the (dimensionless) constant of proportionality be k, equation 3.37 tells us that the intensity of light scattered at an angle θ is

$$I = kI_0\lambda^m r^{-2}\alpha^6(1+\cos^2\theta) \tag{3.39}$$

where we have substituted the fact that $n = -2$. Since λ, r and α all have units of length whilst $\cos^2\theta$ is dimensionless, analysing the units of equation 3.39 gives

$$I = IL^m L^{-2} L^6 = IL^{m-2+6} \tag{3.40}$$

where (with a slight abuse of notation) I represents the units of intensity and L represents the units of length. Since the units of length on both sides of the equation must be equal to each other:

$$0 = m - 2 + 6 \tag{3.41}$$

and so $m = -4$.

13. **Problem**
A glass prism of refractive index $n = 1.40$ has a triangular cross section with two angles of 45°. The prism floats on some mercury with its largest side of length $l = 45.0\,\text{cm}$ facing downwards and a vertical depth of $h = 2.50\,\text{cm}$ submerged.

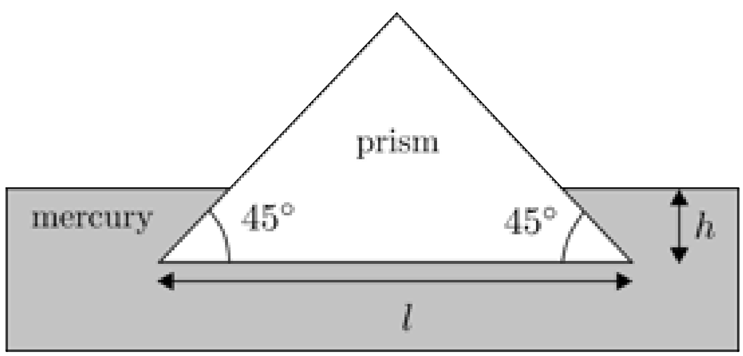


Figure 3.21: A triangular glass prism floating on some mercury.

(a) A monochromatic beam of light, entering the glass parallel to the mercury surface, internally reflects off the bottom face of the prism due to the presence of the mercury. What is the maximum height of the incident beam above the mercury surface such that the beam can leave on the other side of the prism, parallel to the mercury surface?

(b) The prism is then placed on top of a different, clear fluid of the same density and floats. What is the maximum refractive index of the fluid that will allow the light to travel along the same path as in part (a)?

Hint

(a) Add rays to the diagram and label known and unknown values. How do we know that the angle of incidence as the light enters the prism is 45°? Use Snell's law to find the angle of refraction. Can you spot two similar triangles?

(b) You want to find a limiting case for total internal reflection. Can you think of or derive an equation linking θ, n_{liquid} and n_{prism}?

Solution

(a) See Figure 3.22.

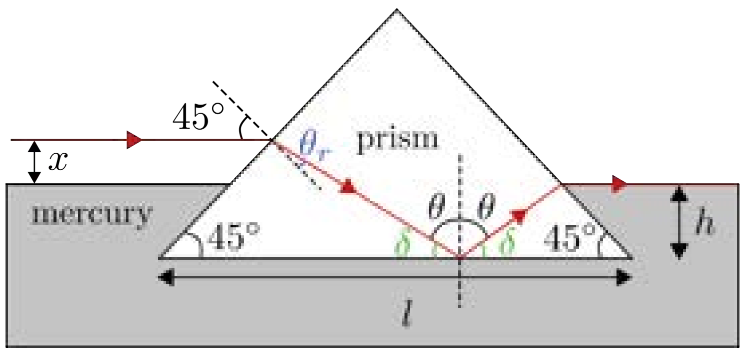


Figure 3.22: A beam of light entering the triangular glass prism parallel to the mercury surface.

This represents the critical case for the light entering the prism, where the reflected light exists and travels along the surface of the mercury.

We want to find x and we are told that $l = 45\,\text{cm}$ and $h = 2.5\,\text{cm}$. From the diagram, we have two similar triangles: one on the right with base b and height h, and another on the left with base $l - b$ and height $h + x$.

We also know, due to the angles in the prism, that the angle of incidence as the light enters the prism and the angle of the light's exit out of the prism are both 45°.

We can use Snell's law to find the angle of refraction θ_r as the light enters the prism. Assuming that the refractive index of air $n_{air} \approx 1$:

$$\sin 45^\circ = n \sin \theta_r \tag{3.42}$$

and so $\theta_r = 30.34^\circ$.

This allows us to figure out angle δ in the left hand triangle:

$$\delta = 180^\circ - 45^\circ - 90^\circ - \theta_r = 14.66^\circ \tag{3.43}$$

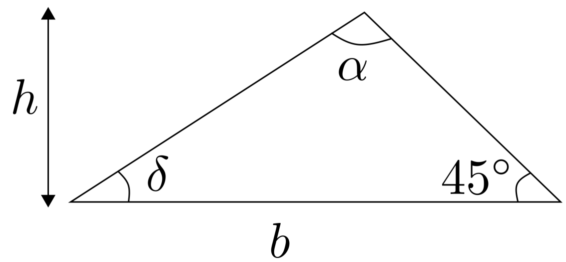


Figure 3.23: A helpful triangle. $\delta = 14.66°$ and $\alpha = 120.34°$.

This, in turn, allows us to figure out the top angle in the right hand triangle as $180° - 45° - \delta = 120.34°$.

Focusing in on the right hand triangle in Figure 3.23, we can use the sine rule to figure out length b:

$$\frac{h}{\sin 45°} = \frac{b}{\sin 120.34°} \tag{3.44}$$

and so

$$b = \frac{h \sin 120.34°}{\sin 45°} \approx 12.06\,\text{cm} \tag{3.45}$$

Since the left hand triangle and right hand triangle are similar:

$$\frac{h+x}{h} = \frac{l-b}{b} \tag{3.46}$$

and so finally

$$x = \frac{h(l-b)}{b} - h \approx 4.33\,\text{cm} \tag{3.47}$$

(b) This is very similar to part (a), with the main difference being that we now want a limiting case for total internal reflection. This occurs when

$$\sin\theta = \frac{n_{liquid}}{n_{prism}} \tag{3.48}$$

and so we wish to find

$$n_{liquid} = n_{prism} \sin\theta = 1.4 \times \sin\theta \tag{3.49}$$

All that is left to do is to find the angle θ. Since we figured out that $\delta = 14.66°$ in the previous part, this means that

$$\theta = 90° - \delta = 75.34° \tag{3.50}$$

and so finally

$$n_{liquid} = 1.4 \times \sin 75.34° \approx 1.35 \tag{3.51}$$

14. **Problem**

On roads, devices known as cat's eyes are used to reflect light from a car's headlights back towards the driver. These are loosely based on how light that enters a cat's eye will be reflected back out in a similar direction, so the eye will often seem to glow at night.

One type of cat's eye is created using a sphere of glass, with a curved mirror over half of its surface. Light entering the sphere is reflected off the mirror and exits the sphere travelling in the exact opposite direction to its direction of travel before entering the sphere (that is, at the same angle to the horizontal).

(a) A beam of light is incident on the surface of the sphere at an angle of $\theta_i = 4.58°$ to the normal of the sphere at that point. If the refractive index is $n = 1.54$, what is the angle through which the incident beam deviates as it is refracted at this first surface? This is the angle between its original direction and its new direction.

(b) Consider an idealised version of the cat's eye, whereby the entire sphere has a refractive index n. The deflection of the beam inside the sphere will depend on this refractive index. θ_i is the angle of incidence of the beam on the sphere and θ_r is the angle of refraction as the beam enters the sphere. What is the total deflection of the beam once it has emerged from the sphere, assuming it only reflects from the mirror once?

(c) Assuming that θ_i and θ_r are small so that the approximation $\sin\theta \approx \theta$ holds, what refractive index would be needed for the beam that has left the sphere to be moving in exactly the opposite direction to the beam before entering the sphere?

Hint

(a) Draw a ray diagram and use Snell's law. Note that the question does not ask for the angle of refraction but the angle of deviation. How are the angles of incidence, refraction and deviation related?

(b) Draw a new ray diagram. Do you already know multiple angles due to the symmetry of the problem? Calculate each deflection separately before adding them together.

(c) If the beam is moving in exactly the opposite direction, then its total deflection (as you calculated in the previous part of the question) is 180°. Use Snell's law and the small angle approximation.

Solution

(a) See Figure 3.24.

Note that the question does not ask for the angle of refraction θ_r, but instead for the angle of deviation, or δ as labelled in the diagram.

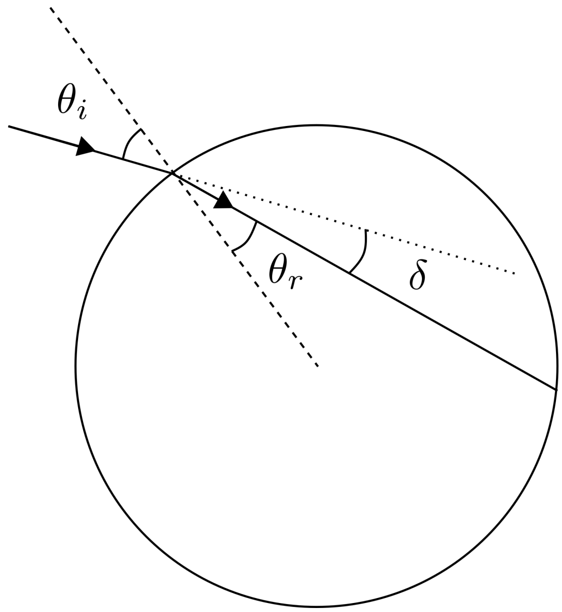


Figure 3.24: The beam of light refracting after hitting the sphere.

Using Snell's law and assuming a refractive index of $n_{air} = 1$ for air:

$$\sin 4.58° = 1.54 \times \sin\theta_r \tag{3.52}$$

This means that

$$\theta_r = \sin^{-1}\left(\frac{\sin 4.58°}{1.54}\right) \approx 2.97° \tag{3.53}$$

Since $\theta_i = \theta_r + \delta$, we can work out that

$$\delta = \theta_i - \theta_r = 4.58° - 2.97° = 1.61° \tag{3.54}$$

(b) See Figure 3.25.

This shows the path of the beam through the cat's eye. Due to the symmetry of the problem, we can add several more angles, as shown in Figure 3.26.

The total deflection of the beam is made up of the three individual reflections, δ_1, δ_2 and δ_3, at each interface:

$$\delta_1 = \theta_i - \theta_r \tag{3.55}$$
$$\delta_2 = 180° - 2\theta_r \tag{3.56}$$
$$\delta_3 = \theta_i - \theta_r \tag{3.57}$$

The total deflection is therefore

$$\delta_1 + \delta_2 + \delta_3 = 180° + 2\theta_i - 4\theta_r \tag{3.58}$$

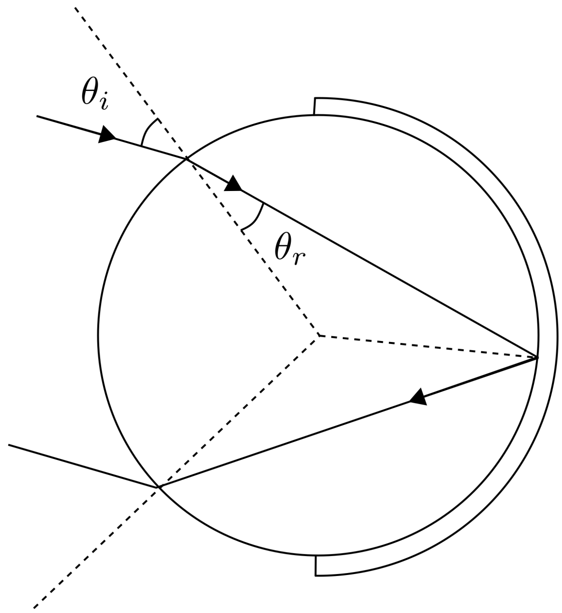


Figure 3.25: The path of the beam of light through the cat's eye.

(c) If the beam is moving in exactly the opposite direction then the total deflection must be equal to 180°. Setting equation 3.58 equal to this value then gives:

$$180^\circ = 180^\circ + 2\theta_i - 4\theta_r \tag{3.59}$$

Solving for θ_i then yields

$$\theta_i = 2\theta_r \tag{3.60}$$

Snell's law (assuming the refractive index of air to be one) is

$$\sin\theta_i = n\sin\theta_r \tag{3.61}$$

and applying the small angle approximation $\sin\theta \approx \theta$ gives

$$\theta_i \approx n\theta_r \tag{3.62}$$

Substituting for θ_i using equation 3.60 then results in

$$2\theta_r \approx n\theta_r \tag{3.63}$$

and so $n = 2$.

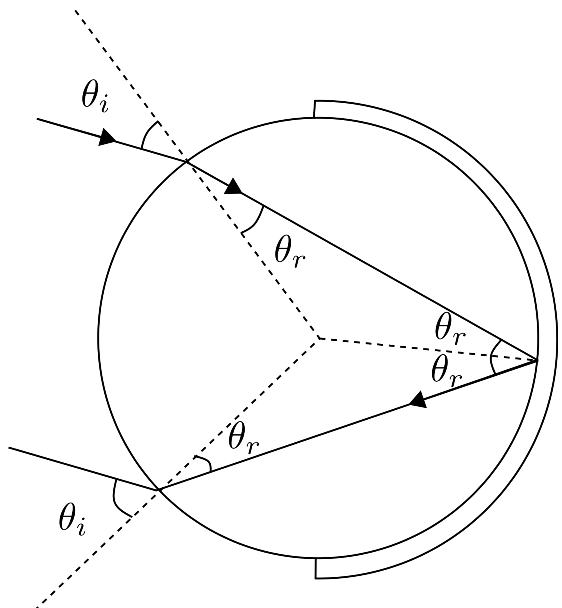


Figure 3.26: The path of the beam of light through the cat's eye, with additional angles labelled.