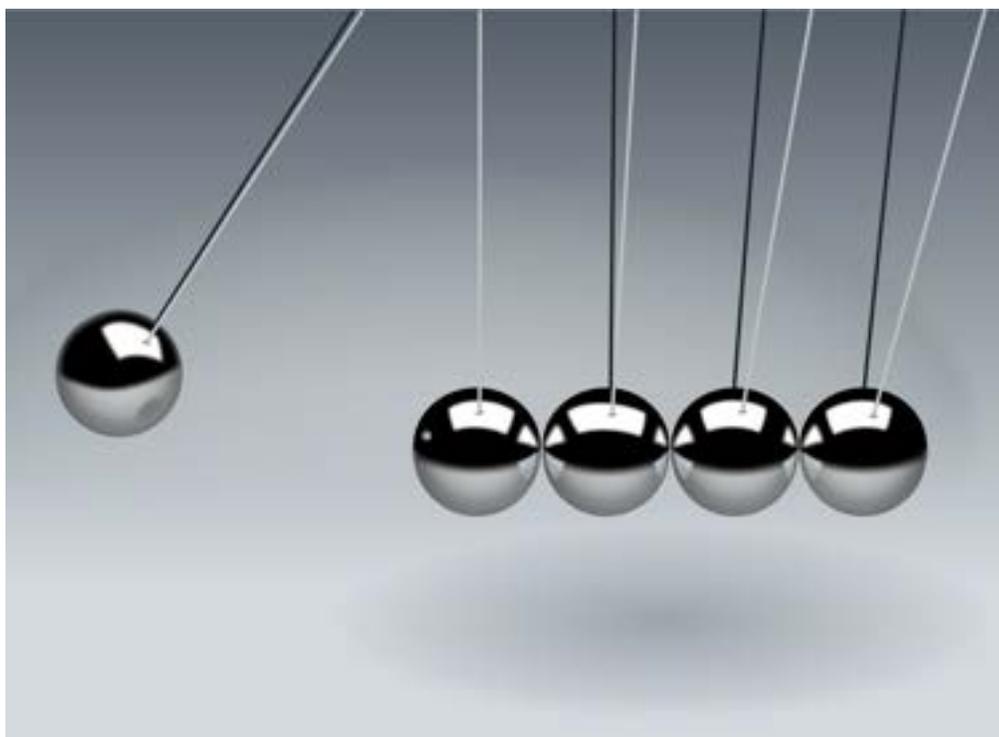# Physics Problems

Workbook 1: Solutions

To help with preparation for the
Physics Aptitude Test (PAT) at the
University of Oxford

DEPARTMENT OF ENGINEERING SCIENCE

Department of Physics

OxfordMaterials

Contents

Chapter 1

Turning words into Maths

1.1 Introductory Problems

1. **Problem**

   A jar contains buttons of four different colours. There are twice as many yellow as green, twice as many red as yellow, and twice as many blue as red. What is the probability of taking from the jar:

   - a blue button;

   - a red button;

   - a yellow button;

   - a green button?

   You may assume that you are only taking one button at a time and replacing it in the jar before selecting the next one.

   Hint

   Assign variables to the numbers of each colour of button and try to convert the information in the question into a set of equations.

   Solution

   Let n_b, n_r, n_y and n_g be the numbers of blue, red, yellow and green buttons respectively. Also let $N = n_b + n_r + n_y + n_g$ be the total number of buttons.

   We need to convert the information given in the question into mathematical statements. The fact that there are "twice as many yellow as green" means that n_y is twice n_g, or equivalently $n_y = 2n_g$. Similarly, "twice as many red as yellow" becomes $n_r = 2n_y$ and "twice as many blue as red" becomes $n_b = 2n_r$.

This gives us:

$$n_y = 2n_g \tag{1.1}$$
$$n_r = 2n_y \tag{1.2}$$
$$n_b = 2n_r. \tag{1.3}$$

Although we don't know the total number of buttons N, equations 1.1–1.3 allow us to figure out the *ratios* of buttons. For example, if there were one green button, then by equation 1.1 there would be two yellow buttons. If there were two yellow buttons, then by equation 1.2 there would be four red buttons. If there were four red buttons, then by equation 1.3 there would be eight blue buttons. The ratio of green : yellow : red : blue is therefore 1:2:4:8.

Once we have the ratio of buttons, we have the probabilities. This is because the probability of picking a blue button is just n_b/N, with similar expressions for the other colours. The probabilities are therefore:

- Blue: $\frac{8}{1+2+4+8} = \frac{8}{15}$
- Red: $\frac{4}{1+2+4+8} = \frac{4}{15}$
- Yellow: $\frac{2}{1+2+4+8} = \frac{2}{15}$
- Green: $\frac{1}{1+2+4+8} = \frac{1}{15}$

2. **Problem**
A gun is designed that can launch a projectile of mass $10\,\text{kg}$ at a speed of $200\,\text{m}\,\text{s}^{-1}$. The gun is placed close to a straight, horizontal railway line and aligned such that the projectile will land further down the line. A small rail car of mass $200\,\text{kg}$ and travelling at a speed of $100\,\text{m}\,\text{s}^{-1}$ passes the gun just as it is fired. Assuming the gun and the car are at the same level, at what angle upwards must the projectile be fired so that it lands in the rail car?

Hint
Draw a clear diagram to set up the problem. Do you need all the information that the question gives you, or are some pieces of information irrelevant?

Solution
For projectile/SUVAT questions such as these it is important to look for information that is hidden in the question. For example, there is a vertical acceleration of $a = -g = -9.81\,\text{m}\,\text{s}^{-2}$ implicitly present in the problem, even though it isn't mentioned. You are also expected to assume there is no air resistance, so there will be no horizontal accelerations of any kind. Likewise, although you don't have to do this explicitly, it is worth thinking about what the following pieces of information tell us:

- "Passes the gun just as it's fired": we can set $t = 0$ at this point and then use the same t coordinate for both the rail car and the projectile.

- "Assuming the gun and the car are at the same level": we don't have to worry about any vertical displacement or offset between the two objects.

Finally, given that this exists in the world of projectiles, in which objects are in free-fall, the masses of the objects are irrelevant. Their inclusion in the question is a red herring!

A clear diagram is often the best way to lay out all of the information given to you in a problem – see Figure 1.1.
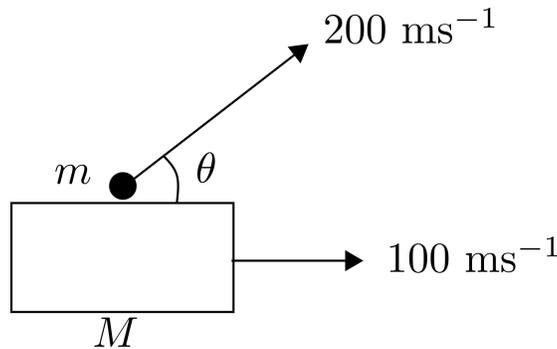

Figure 1.1: A diagram showing the gun firing a projectile of mass m at an angle θ to the horizontal and at a speed of $200\,\mathrm{m\,s^{-1}}$ at the same time as the rail car of mass M passes by at $100\,\mathrm{m\,s^{-1}}$.

There are two points at which the projectile and the car are in the same position:

- When the projectile is launched (let this be $t = 0$)
- When the projectile lands in the car (let this be $t = t$)

Upon landing the two objects therefore have the same horizontal displacement, $s_p = s_c$. It is worth adopting straightforward notation: any quantity with a subscript p refers to the projectile, and any quantity with a subscript c refers to the car.

The horizontal acceleration of the rail car is zero, and the vertical acceleration of the projectile, if we take upward to be positive, is $a_p = -9.81\,\mathrm{m\,s^{-2}}$. Note how we've managed to "pull" a lot of information out of the question that we weren't explicitly given.

Let θ be the angle above the horizontal at which the projectile is launched. The horizontal speed of the projectile is then $u_p = 200\cos\theta\,\mathrm{m\,s^{-1}}$ and the

horizontal speed of the car is $u_c = 100\,\mathrm{m\,s^{-1}}$. Given that they must both travel the same distance in the same time for one to land in the other:

$$s_p = s_c \tag{1.4}$$
$$u_p t = u_c t \tag{1.5}$$
$$u_p = u_c \tag{1.6}$$
$$\tag{1.7}$$

Substituting $u_p = 200\cos\theta\,\mathrm{ms^{-1}}$ and $u_c = 100\,\mathrm{ms^{-1}}$ and simplifying:

$$200\cos\theta = 100 \tag{1.8}$$
$$\cos\theta = \frac{100}{200} \tag{1.9}$$
$$\theta = \cos^{-1}\left(\frac{1}{2}\right)\left(\tag{1.10}$$
$$\theta = 60° \tag{1.11}$$

Notice that the way we have solved this allows us to appreciate the general fact that the horizontal velocities of the car and the projectile must be the same no matter what the other parameters of the question. It is worth thinking about why this makes sense: assuming no air resistance there is no horizontal acceleration in the question, so for the two objects to both start and end at the same horizontal displacement they must always be travelling at the same horizontal velocity.

3. **Problem**
   The planet Pluto (radius 1180 km) is populated by three species of purple caterpillar. Studies have established the following facts:

   - A line of 5 mauve caterpillars is as long as a line of 7 violet caterpillars.

   - A line of 3 lavender caterpillars and 1 mauve caterpillar is as long as a line of 8 violet caterpillars.

   - A line of 5 lavender caterpillars, 5 mauve caterpillars and 2 violet caterpillars is 1 m long in total.

   - A lavender caterpillar takes 10 s to crawl the length of a violet caterpillar.

   - Violet and mauve caterpillars both crawl twice as fast as lavender caterpillars.

   How long would it take a mauve caterpillar to crawl around the equator of Pluto?

Hint

Translate each fact into a mathematical statement, making sure you adopt sensible notation. By isolating the information you need, you should end up with a set of three simultaneous equations which you can solve. Don't be put off if you end up with unfriendly fractions. How is the caterpillar's speed related to the time it takes to crawl around Pluto's equator?

Solution

To answer this question we need to know two things: the distance around Pluto's equator and the crawling speed of a mauve caterpillar. Since we are provided with the radius of Pluto in the question, we can calculate the distance via the formula $2\pi r$. The crawling speed of a mauve caterpillar, however, is considerably more difficult to find.

The first thing to do is to replace as many words with symbols as possible. We could use 'm' to represent the length of a mauve caterpillar, 'v' to represent the length of a violet caterpillar, and 'l' to represent the length of a lavender caterpillar, but then what symbol would we use to represent the crawling speed of each? It will be easier to use standard symbols for quantities, and use subscripts to indicate which caterpillar we are writing about. Thus, the length of a mauve caterpillar will be l_m and the crawling speed of a lavender caterpillar will be v_l.

Our five pieces of information then become:

(a) $5l_m = 7l_v$

(b) $3l_l + l_m = 8l_v$

(c) $5l_l + 5l_m + 2l_v = 1$

(d) $v_l = l_v/10$

(e) $v_v = 2v_l$ and $v_m = 2v_l$.

Where we are told something is "as long as" something else, that is the same as saying they are equal. "Twice" means a factor of two. The fourth fact, which – unlike the other facts – contained information about time taken, has now become about speed and distance (using the fact that speed is distance divided by time taken), to make it more similar to the other four points.

Given that we are trying to find the speed of a mauve caterpillar, it makes sense to start with the second half of the fifth fact, $v_m = 2v_l$, and combine it with the fourth fact:

$$v_m = 2v_l \tag{1.12}$$

$$v_m = 2\left(\frac{l_v}{10}\right)\Bigg(\tag{1.13}$$

So in order to find the speed of a mauve caterpillar v_m, we need to find l_v, the length of a violet caterpillar.

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Facts (a)–(c) all contain a term in l_v. Looking at these facts, they represent three equations with three unknowns (l_m, l_v and l_l) and so are a set of simultaneous equations we can solve for l_v.

Rearranging fact (a) gives that $l_m = 7l_v/5$, and substituting this into fact (b) gives:

$$3l_l + \frac{7l_v}{5} = 8l_v \tag{1.14}$$

which can be solved for an expression for l_l in terms of l_v:

$$l_l = \frac{33}{15}l_v \tag{1.15}$$

You should not be put off if, in your solution, you encounter some more unusual fractions than you might expect in other exams. Substituting our expressions for l_m and l_l into fact (c) then gives

$$5\left(\frac{33}{15}l_v\right) + 5\left(\frac{7}{5}l_v\right) + 2l_v = 1 \tag{1.16}$$

Simplifying:

$$11l_v + 7l_v + 2l_v = 20l_v = 1 \tag{1.17}$$

and we can rearrange this for l_v to find that

$$l_v = \frac{1}{20} = 0.05\,\text{m} \tag{1.18}$$

Since, by equation 1.13, $v_m = 2l_v/10$, this means that the speed of the mauve caterpillar v_m is

$$v_m = \frac{2 \times 0.05}{10} = 0.01\,\text{m}\,\text{s}^{-1} \tag{1.19}$$

Now that we have v_m we can determine the final answer. The circumference of Pluto is $C = 2\pi r$, where r is given to us in the question as $r = 1180\,\text{km}$. If we didn't have a calculator, estimating $\pi \approx 3$ gives:

$$C \approx 2 \times 3 \times 1200 = 7200\,\text{km} \tag{1.20}$$

Note that we've approximated $1180\,\text{km}$ as $1200\,\text{km}$, but we also "underestimated" the value of π so we've got a reasonable answer. The time take to crawl is then given by

$$t = \frac{C}{v_m} = \frac{7200 \times 10^3}{1 \times 10^{-2}} = 7.2 \times 10^8 \text{ seconds} \approx 22.8\,\text{years} \tag{1.21}$$

If we were using a calculator we'd get $t = 7.4 \times 10^8$ s $\approx 23.5\,\text{years}$, which is less than 3% different.

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1.2 Further Problems

4. **Problem**

The drag force F on a sphere is related to the radius of the sphere, r, the velocity of the sphere, v, and the coefficient of viscosity of the fluid the drop is falling through, η, by the formula

$$F = kr^x\eta^y v^z \tag{1.22}$$

where k is a dimensionless constant, and x, y, and z are integers. By considering the units of the equation, work out the values of x, y, and z. (The coefficient of viscosity has units of $\mathrm{kg\,m^{-1}s^{-1}}$.)

Hint

Essentially, this problem boils down to solving the equation:

$$\mathrm{kg\,m\,s^{-2}} = [\mathrm{m}]^x[\mathrm{kg\,m^{-1}s^{-1}}]^y[\mathrm{m\,s^{-1}}]^z \tag{1.23}$$

Remember that $(ab)^x = a^x b^x$ and note that there is only one kilogram term on either side of the equals sign.

Solution

The units must be the same on both sides of the equation. Given that k is dimensionless:

$$F = kr^x\eta^y v^z \tag{1.24}$$
$$N = [m]^x[\mathrm{kg\,m^{-1}s^{-1}}]^y[\mathrm{ms^{-1}}]^z \tag{1.25}$$
$$\mathrm{kg\,m\,s^{-2}} = [\mathrm{m}^x][\mathrm{kg}^y\mathrm{m}^{-y}\mathrm{s}^{-y}][\mathrm{m}^z\mathrm{s}^{-z}] \tag{1.26}$$
$$\mathrm{kg\,m\,s^{-2}} = \mathrm{m}^x\mathrm{kg}^y\mathrm{m}^{-y}\mathrm{s}^{-y}\mathrm{m}^z\mathrm{s}^{-z} \tag{1.27}$$

Given that there is only one kilogram term on either side of the equation, y must be equal to 1.

$$\mathrm{kg\,m\,s^{-2}} = \mathrm{m}^x\mathrm{kg\,m^{-1}s^{-1}}\mathrm{m}^z\mathrm{s}^{-z} \tag{1.28}$$

If we add up the indices, this gives us:

$$\mathrm{kg\,m\,s^{-2}} = \mathrm{kg\,m}^{x+z-1}\mathrm{s}^{-z-1} \tag{1.29}$$

which then gives us $1 = x + z - 1$ and $-2 = -z - 1$. Solving these couple of equations gives $z = 1$ and $x = 1$ and our final equation is simply

$$F = kr\eta v \tag{1.30}$$

5. **Problem**

A radioactive sample contains two different isotopes, A and B. A has a half-life of 3 days, and B has a half-life of 6 days. Initially in the sample there are twice as many atoms of A as of B. At what time will the ratio of the number of atoms of A to B be reversed?

Hint

There are two factors of two in the question: there are twice as many atoms of A than B, so $N_A = 2N_B$, and the half-life of B is twice that of A. Use $N = N_0 e^{-\lambda t}$ and remember that you are looking for the situation to be reversed, i.e. $N_A = N_B/2$.

Solution

Using the fact that $N = N_0 e^{\lambda t}$:

$$N_A = N_{A_0} e^{-\lambda_A t} \tag{1.31}$$

$$N_B = N_{B_0} e^{-\lambda_B t} \tag{1.32}$$

Initially $N_{A_0} = 2N_{B_0}$ and we want to know how long it takes for $N_A = N_B/2$. The relationship between the half-life $t_{\frac{1}{2}}$ of a substance and its decay constant λ is

$$t_{\frac{1}{2}} = \frac{\ln 2}{\lambda} \tag{1.33}$$

Because the half-life of A is half the half-life of B, the time constant $\lambda_A = 2\lambda_B$.

We are looking for the time t when

$$N_A = \frac{N_B}{2} \tag{1.34}$$

Substituting the expressions for N_A and N_B gives

$$N_{A_0} e^{-\lambda_A t} = \frac{N_{B_0} e^{-\lambda_B t}}{2} \tag{1.35}$$

But $N_{A0} = 2N_{B0}$ and $\lambda_A = 2\lambda_B$ so

$$2N_{B_0} e^{-2\lambda_B t} = \frac{N_{B_0} e^{-\lambda_B t}}{2} \tag{1.36}$$

Multiplying both sides by 2 and dividing through by N_{B0}:

$$4e^{-2\lambda_B t} = e^{-\lambda_B t} \tag{1.37}$$

Taking natural logs of both sides and simplifying using standard properties of logs:

$$\ln(4e^{-2\lambda_B t}) = \ln(e^{-\lambda_B t}) \tag{1.38}$$

$$\ln(4) - 2\lambda_B t = -\lambda_B t \tag{1.39}$$

$$\lambda_B t = \ln(4) \tag{1.40}$$

$$t = \frac{\ln(4)}{\lambda_B} \tag{1.41}$$

We can rearrange equation 1.33 to relate the decay constant λ_B to the half-life of B t_B. Working in days, this gives a final value for t of

$$t = \frac{\ln 4}{\lambda_B} = \ln 4 \times \frac{t_B}{\ln 2} = 2\ln 2 \times \frac{6 \text{ days}}{\ln 2} = 12 \text{ days} \qquad (1.42)$$

where we have used the fact that $\ln 4 = \ln 2^2 = 2\ln 2$.

6. **Problem**

A snooker ball must be 5.175 cm in diameter to within an uncertainty of ± 0.127 mm. The Earth is 6371 km in radius and its highest mountain above sea level, Mount Everest, is 8848 m. Which is smoother, a snooker ball or the Earth? [Note: do we know everything to the same level of accuracy?]

Hint

Think about percentage error when approaching this question. Make sure you convert between millimetres, centimetres, metres and kilometres as there are a lot of different units at play.

Solution

As a fraction of the Earth's radius, R, Mount Everest's 8848 m height, h, is

$$\frac{h}{R} = \frac{8848}{6371 \times 10^3} = 1.389 \times 10^{-3} \qquad (1.43)$$

or 0.1389%. For the snooker ball, the ratio is

$$\frac{h}{R} = \frac{0.127 \times 10^{-3}}{5.175 \times 10^{-2}} = 2.45 \times 10^{-3} \qquad (1.44)$$

or 0.245%. Therefore the Earth is smoother than a snooker ball.

If we take the tolerance of a snooker ball and extrapolate it to the size of the Earth,

$$h = \frac{0.127 \times 10^{-3}}{5.175 \times 10^{-2}} \times 6371 \times 10^3 = 15\,640 \text{ m} \qquad (1.45)$$

If a snooker ball were enlarged to the size of the Earth, the largest mountain would be 15640 m high, nearly twice the height of Mount Everest.

1.3 Extension Problems

7. **Problem**

A ball of mass $0.1\,\mathrm{kg}$ bounces on a hard surface. Every time it hits the floor, it loses a quarter of its kinetic energy. If the ball is released from a height of $1.00\,\mathrm{m}$, after how many bounces will the ball bounce no higher than $0.25\,\mathrm{m}$?

Hint

Remember that gravitational potential energy depends linearly on height above the ground. After each bounce, 75% of the energy remains, so how many times do you need to take 75% of 75% of 75% ...etc. to have only 25% remaining?

Solution

Firstly, it is worth noticing that the principle of conservation of energy (whereby PE + KE = constant) does not apply in this case since the ball is losing energy after every bounce.

Let the ball start at an initial height h_0. Its initial gravitational potential energy is then GPE $= mgh_0$. At the moment before it bounces for the first time, this potential energy has been entirely converted into kinetic energy, such that $KE_0 = mgh_0$. However, after its first bounce the ball has lost a quarter of its kinetic energy, such that

$$KE_1 = \frac{3}{4}mgh_0 = mgh_1 \qquad (1.46)$$

where h_1 is the height the ball reaches after the first bounce and KE_1 is its kinetic energy after the first bounce. This can be simplified into an equation linking the initial and final heights:

$$\frac{3}{4}h_0 = h_1 \qquad (1.47)$$

and so we see that the mass of the ball makes no difference. If the question were changed so that the ball were $10\,\mathrm{kg}$ or $2718\,\mathrm{kg}$ but everything else remained the same, the answer would still be the same.

Since $GPE = mgh$, the height, h that the ball reaches depends linearly on the gravitational potential energy. Therefore it will bounce no higher than one quarter of its starting height when it has no more than one quarter of its gravitational potential energy left.

Given that the ball retains 3/4 of its energy during each bounce, the height it reaches after n bounces is

$$h_n = \left(\frac{3}{4}\right)^n \times h_0 \qquad (1.48)$$

By setting $h_0 = 1$ (since the ball is released from a height of 1.00 m) we can then solve to find when $h_n \leq 0.25$:

$$0.25 \leq (0.75)^n \tag{1.49}$$

$$\ln 0.25 \leq \ln 0.75^n \tag{1.50}$$

$$\ln 0.25 \leq n \ln 0.75 \tag{1.51}$$

$$n \geq 4.81 \tag{1.52}$$

Therefore the ball will reach above $0.25\,\mathrm{m}$ after its fourth bounce, and below $0.25\,\mathrm{m}$ after its fifth bounce.

8. **Problem**

You want to make a snowman out of modelling clay. The snowman consists of two spheres, where one sphere has a radius r, and the other has a radius $2r$. The modelling clay comes in the form of a cylinder with radius $r/2$. What length of modelling clay is required to make the snowman?

Hint

If the larger sphere has twice the radius, it has eight (2^3) times the volume. Remember that the formula for the volume of a sphere is $V_S = \frac{4}{3}\pi r^3$ and of a cylinder is $V_C = \pi r^2 h$.

Solution

The volume of the smaller sphere, V_S, is $V_S = \frac{4}{3}\pi r^3$. The volume of the larger sphere, V_L, being twice the radius, is eight times larger: $V_L = 8V_S = \frac{32}{3}\pi r^3$. The total volume of the two spheres is then

$$V_T = \left(\frac{4}{3} + \frac{32}{3}\right)\pi r^3 = 12\pi r^3 \tag{1.53}$$

The radius of the cylinder is $r_c = r/2$. The volume of the cylinder is therefore

$$V_C = \pi \left(\frac{r}{2}\right)^2 h = \frac{\pi r^2 h}{4} \tag{1.54}$$

Given that the two volumes are equal, we can solve for the length h in terms of r:

$$V_C = V_T \tag{1.55}$$

$$\frac{\pi r^2 h}{4} = 12\pi r^3 \tag{1.56}$$

$$\pi r^2 h = 48\pi r^3 \tag{1.57}$$

$$h = 48r \tag{1.58}$$

Therefore the required cylinder of modelling clay will have a length 48 times the radius of the smaller sphere that makes up the snowman.

9. **Problem**

   A mass m is hung from a spring with a spring constant of k. When set into motion, the mass oscillates with a period

   $$T = 2\pi\sqrt{\frac{m}{k}}. \tag{1.59}$$

   Using another identical spring:

   (a) What would be the period of oscillation of the mass if it were taken to a planet with a gravitational field strength of $2g$?

   (b) What would be the period of oscillation of the mass if it were hung from the two springs connected end-to-end (in series)?

   (c) What would be the period of oscillation of the mass if it were hung from the two springs connected side-by-side (in parallel)?

   Hint

   (a) What changes?

   (b) In this case, each spring experiences the same force of the weight of the mass. What effect does this have on k and therefore T?

   (c) In this case, each spring experiences half the force of the weight of the mass. What effect does this have on k and therefore T?

   Solution

   The period of a mass-spring system is $T = 2\pi\sqrt{m/k}$.

   (a) There are no g terms in the expression for the period of oscillation, and g doesn't affect m or k, so taking the system to a different planet makes no difference. (This is how astronauts weigh themselves on the International Space Station: they sit on a calibrated spring and measure the period of their oscillation.)

   (b) With the springs connected end-to-end, the same force acts on both springs, and the extension of the system is twice what it was before. Therefore the spring constant is half what it was previously. As $T \propto \sqrt{1/k}$, the system's new period is $T\sqrt{2} \approx 1.4T$.

   For a more mathematical approach, consider Figure 1.2.

   The same force (the mass' weight) acts on both springs. This force is given by Hooke's law: $F = kx$, where x is the spring's extension. This means that

   $$x = \frac{F}{k} \tag{1.60}$$

   The total extension x_T is just the sum of the two springs' individual extensions:

   $$x_T = x_1 + x_2 \tag{1.61}$$

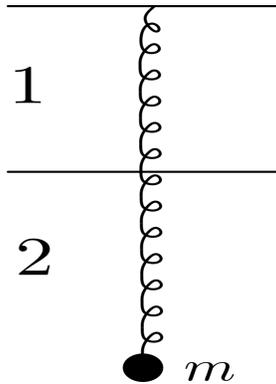 $$\frac{F}{k_T} = \frac{F}{k_1} + \frac{F}{k_2} \tag{1.62}$$

Figure 1.2: A diagram showing the mass hung from the two springs connected in series.

where k_T is the total effective spring constant of the two springs in series. However, since both springs are identical, $k_1 = k_2$. Dividing through by F gives

$$\frac{1}{k_T} = \frac{1}{k} + \frac{1}{k} \rightarrow k_T = \frac{k}{2} \tag{1.63}$$

This tells us that, when springs are connected in series, the effective spring constant gets smaller. Finally, since $T \propto \sqrt{1/k}$ and the spring constant for the springs in series $k_T = k/2$, the period of oscillation for springs in series T_s is:

$$T_s \propto \sqrt{\frac{1}{k_T}} \tag{1.64}$$

$$T_s \propto \sqrt{\frac{2}{k}} \tag{1.65}$$

$$T_s \propto \sqrt{\frac{1}{k}} \times \sqrt{2} \tag{1.66}$$

$$T_s = T\sqrt{2} \tag{1.67}$$

Hence, when the springs are connected in series, the period of oscillation increases.

(c) With the springs connected side-by-side, half the force acts on each spring, and the extension of the system is half what it was before. Therefore the spring constant is twice what it was previously. As $T \propto \sqrt{1/k}$, the system's new period is $T/\sqrt{2}$ or $\approx 0.71T$.

For a more mathematical approach, consider Figure 1.3.

Whereas in part (b) the force acting on each spring was the same, in this situation now it is the extension x of each spring which must be the same.
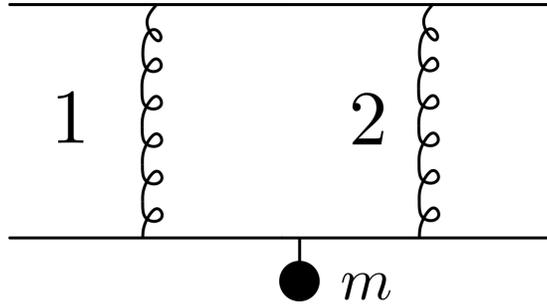
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Figure 1.3: A diagram showing the mass hung from the two springs connected in parallel.

The total force on the springs is just the force on each individual spring: $F_T = F_1 + F_2$. Using Hooke's law:

$$k_T x = k_1 x_1 + k_2 x_2 \tag{1.68}$$

where this time k_T is the total effective spring constant of the two springs in parallel. Since the extensions of each spring are identical, $x_1 = x_2 = x$. This means that

$$k_T = k_1 + k_2 \tag{1.69}$$

But the springs are identical, so $k_1 = k_2$ as before. Hence

$$k_T = 2k \tag{1.70}$$

This shows that, when the springs are connected in parallel, the effective spring constant increases.

Using the same tactic as in part (b), since $T \propto \sqrt{1/k}$, the period of oscillation for springs in parallel T_p is:

$$T_p \propto \sqrt{\frac{1}{k_T}} \tag{1.71}$$

$$T_p \propto \sqrt{\frac{1}{2k}} \tag{1.72}$$

$$T_p \propto \sqrt{\frac{1}{k}} \times \frac{1}{\sqrt{2}} \tag{1.73}$$

$$T_p = \frac{T}{\sqrt{2}} \tag{1.74}$$

Hence, when the springs are connected in parallel, the period of oscillation decreases.

10. **Problem**

A ball is thrown at an angle of 30° up from the horizontal, at a speed of $10\,\mathrm{m\,s^{-1}}$, off the top of a cliff which is $10\,\mathrm{m}$ high above a flat beach. How long does it take for the ball to hit the beach below? You may assume that the acceleration due to gravity is $10\,\mathrm{m\,s^{-2}}$, and that air resistance can be neglected.

Hint

Do we need to consider both horizontal and vertical motion, or can we only focus on one? Remember that you can calculate both the negative and positive displacement using SUVAT equations.

Solution
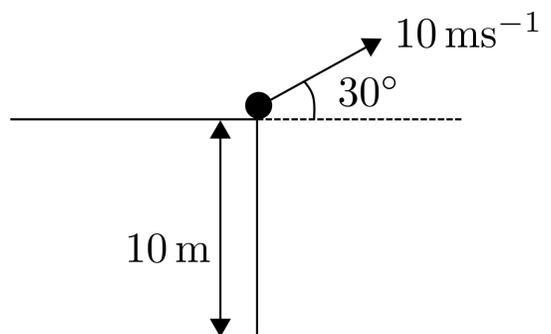
Figure 1.4: A diagram showing a ball being thrown at $10\,\mathrm{m\,s^{-1}}$ at an angle of 30° to the horizontal off the top of a cliff of height $10\,\mathrm{m}$.

We're not interested in any horizontal movement, so we will only work vertically. Our initial horizontal velocity is $u = 10\sin(30°)\mathrm{m\,s^{-1}} = 5\mathrm{m\,s^{-1}}$ and the vertical acceleration is $a = g = -10\,\mathrm{m\,s^{-2}}$. The final displacement is ten metres below the launch point, so $s = -10\,\mathrm{m}$. Using a standard SUVAT equation:

$$s = ut + \frac{1}{2}at^2 \tag{1.75}$$

$$-10 = 5t + \frac{1}{2} \times (-10)t^2 \tag{1.76}$$

This can be rearranged into the form $5t^2 - 5t - 10 = 0$ which we can simplify and factorise to find two possible values of t:

$$(t-2)(t+1) = 0 \rightarrow t = 2 \text{ or } t = -1 \tag{1.77}$$

Only the positive solution makes physical sense, so that is our answer:

$$t = 2\,\mathrm{s} \tag{1.78}$$

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The negative solution corresponds to when the ball would have been launched from the ground if we were to follow its parabolic trajectory backwards.

11. **Problem**

I walk down a long street at $1\,\mathrm{m\,s^{-1}}$ for an hour. During this time I count the number of trams that pass me by. Knowing that they follow a regular timetable in both directions I am initially surprised to note that only 15 trams overtake me, whereas 20 passed me head-on. What is the average speed of the trams?

Hint

There is more than one way to approach this problem. What is the time gap between consecutive trams overtaking me (and consecutive trams passing me head-on), and how far do I walk in this time? Can you work out the speed from this information? Alternatively, consider transferring into the frame of reference in which the person is stationary. This problem is related to the Doppler effect: thinking in terms of waves could be helpful here.

Solution 1

The trams overtaking me are separated by $60/15 = 4$ minutes, whilst the trams approaching head-on are separated by $60/20 = 3$ minutes.

Let the separation between the trams be the distance d. The overtaking trams must travel this distance plus the distance I walk in between each tram. This means that the overtaking trams must travel $d + 240\,\mathrm{m}$ in 4 minutes (since $240\,\mathrm{m}$ is how far I've walked in the 4 minutes $= 240$ seconds between trams).

Similarly, the head-on trams travel $d - 180\,\mathrm{m}$ in 3 minutes.

Since each tram travels at the same speed, the difference between these two distances is the distance the trams travel in one minute. That is,

$$d + 240 - (d - 180) = 420\mathrm{m} \tag{1.79}$$

is the distance the trams travel in 1 minute. This therefore means that the trams travel at a speed of $420/60 = 7\,\mathrm{m\,s^{-1}}$.

Solution 2

In my frame of reference, 20 trams pass by me in one direction with a speed of $v + 1$, whereas 15 trams pass by me in the other direction with a speed of $v - 1$. It may help to think in terms of the wave equation:

$$\text{wave speed} = \text{frequency} \times \text{wavelength} \tag{1.80}$$

Since the distance (e.g. wavelength) between the trams is constant, we see that the speed is directly proportional to the frequency.

This means that $(v+1)$ is directly proportional to 20, and $(v-1)$ is directly proportional to 15 via the same proportionality constant. Explicitly:

$$v + 1 \propto 20 \rightarrow v + 1 = 20k \tag{1.81}$$

$$v - 1 \propto 15 \rightarrow v - 1 = 15k \tag{1.82}$$

Solving these simultaneous equations we see that

$$\frac{20}{15} = \frac{(v+1)}{v-1} \tag{1.83}$$

which leads to $v = 7\,\mathrm{m\,s^{-1}}$.

12. **Problem**

Hayley and Rob offer to paint the outside of our house. Hayley claims she can do the job in 2 days (working continuously). Rob says he'll complete the job in 3 days. If they are both hired to work together, how long should it take?

Hint

From the information given, consider how much each person can paint in one day.

Solution

This is a 'rate' problem, so as the hint suggests it helps to figure out how much each person can paint in one day. 2 Robs would take 1.5 days and 2 Hayleys would take 1 day, so we're looking for an answer in between. Hayley can do $\frac{1}{2}$ of a house per day, whereas Rob can do $\frac{1}{3}$ of a house per day. This means that together they do $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ of a house per day. Hence it will take them $\frac{6}{5} = 1.2$ days to paint the house.

Chapter 2

Setting up physics problems

2.1 Introductory Problems

1. **Problem**
   Rectangle ABCD has an area of $120\,\text{cm}^2$. Find the area of the shaded part.
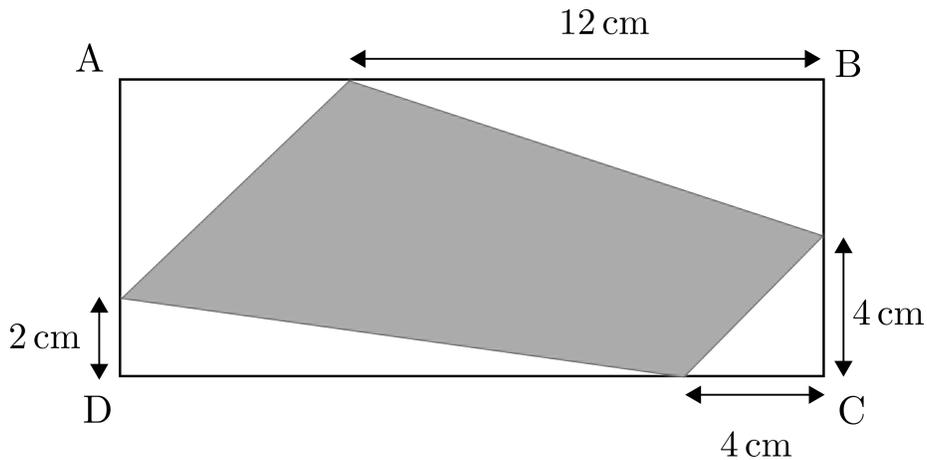
Figure 2.1: Shaded rectangle ABCD

Hint
Try breaking the shaded area up into more familiar shapes. For every unshaded triangle, can you find an equivalent shaded triangle?

Solution
See Figure 2.2. The central rectangle is $2\,\text{cm} \times 8\,\text{cm}$ so has an area of $16\,\text{cm}^2$. This leaves four triangles, which now split the remaining area of

$120 - 16 = 104\,\text{cm}^2$ in half. The area of the shaded region is therefore

$$\frac{104}{2} + 16 = 68\,\text{cm}^2 \tag{2.1}$$
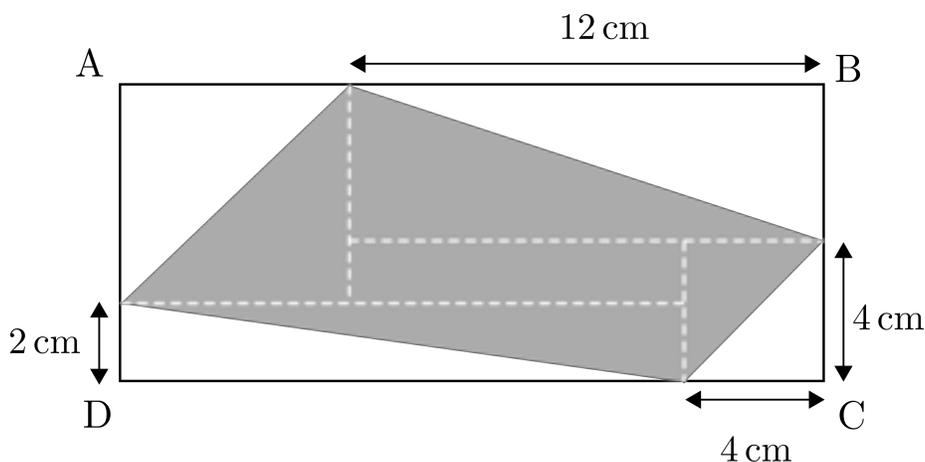
Figure 2.2: Shaded rectangle ABCD split into useful areas

2. **Problem**

A ball is at rest at the top of a frictionless hill. It is then given a slight nudge and speeds up to $4\,\text{ms}^{-1}$ at the bottom of the hill. If the ball had a speed of $3\,\text{ms}^{-1}$ at the top of the hill (instead of being stationary), what would its speed now be at the bottom?

Hint

Why do you think the mass of the ball has not been specified? Have you identified which conservation law is useful here? Do not substitute numbers too early. Do you need to evaluate the height?

Solution

The key principle here is the conservation of energy: the total energy of the ball (made up of its gravitational potential energy and its kinetic energy) must be constant. In other words, the ball's initial and final energies must be equal to each other.

Let the hill have a height h. There are two scenarios to consider: one when the ball starts off stationary and one when the ball starts off at $3\,\text{ms}^{-1}$.

- If the ball starts off stationary, then its initial energy is simply mgh. Its final energy is all kinetic and is $mv^2/2$ where $v = 4\,\text{ms}^{-1}$ is its speed at the bottom of the hill. By the conservation of energy, these two must be equal to each other:

$$mgh = \frac{1}{2}mv^2 \tag{2.2}$$

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- If the ball starts off at $3\,\mathrm{ms}^{-1}$ then its initial energy is $mgh + mu^2/2$ where $u = 3\,\mathrm{ms}^{-1}$. Its final energy at the bottom of the hill is all kinetic and is simply $mw^2/2$ where w is its speed at the bottom of the hill. By the conservation of energy:

$$mgh + \frac{1}{2}mu^2 = \frac{1}{2}mw^2 \qquad (2.3)$$

By substituting equation 2.2 from the stationary case into equation 2.3, we have that

$$\frac{1}{2}mv^2 + \frac{1}{2}mu^2 = \frac{1}{2}m(v^2 + u^2) = \frac{1}{2}mw^2 \qquad (2.4)$$

This leads to $w^2 = u^2 + v^2$ and substituting $u = 3\,\mathrm{ms}^{-1}$ and $v = 4\,\mathrm{ms}^{-1}$ leads to a final answer of $w = 5\,\mathrm{ms}^{-1}$.

3. **Problem**

A $10 \times 10 \times 10$ cube is constructed from a thousand unit cubes. How many of the unit cubes have at least one face on the surface of the larger cube?

Hint

How will you deal with the edges and corners? Could you ask a different question that is almost equivalent to give the answer in one step?

Solution

The question asks how many $1 \times 1 \times 1$ cubes are on the outside of a $10 \times 10 \times 10$ megacube? Let us instead ask the opposite question – how many cubes are on the inside?

The inside will be an $8 \times 8 \times 8$ cube, so is made of 512 individual cubes. Therefore the number of cubes on the outside must be the remainder, i.e. $1000 - 512 = 488$ cubes have at least one face on the surface of the larger cube.

4. **Problem**

A bartender pours $100\,\mathrm{ml}$ of tonic water into one glass and $100\,\mathrm{ml}$ of gin into another. He then takes a $30\,\mathrm{ml}$ shot glass and scoops a shot of gin into the tonic and gives it a good mix. He then takes the shot glass and transfers $30\,\mathrm{ml}$ of the mix back into the gin glass. Is there more tonic in the gin glass, or more gin in the tonic glass?

Hint

What is the final volume of liquid in each glass? Although numbers are given here, they are not necessary to solve this.

Solution

It is possible to solve this mathematically, but there's no need to calculate anything explicitly. Each glass starts and ends with the same total volume of liquid, so any gin that is missing from one glass (and in the other) must have been replaced by tonic from the other and vice versa. So there is an equal amount of tonic in the gin glass as there is gin in the tonic glass.

5. **Problem**

A race has 2021 entrants, all numbered from 1 to 2021 at random. What is the probability that the first three runners to cross the finish line are numbered in ascending order?

Hint

Would it make a difference if there were only 2020 runners? It may be useful not to focus on the numbers: why not label the winners A, B and C?

Solution

The numbers themselves of the runners crossing the finish line are not important: whatever they are you can always assign a label to the first three runners across the line at random. Let's call them A, B and C.

There are only 6 possible permutations in which they could have crossed the line: ABC, ACB, BAC, BCA, CAB or CBA. We assume each of these is equally likely (a 1 in 6 chance), and only one of these can correspond to the numbers being in ascending order. So the probability that the first three runners to cross the finish line are numbers in ascending order is 1/6.

6. **Problem**

An ant is at one corner of a cube of side length a. What is the minimum distance it must travel to reach the far corner of the cube?

Hint

Can you turn this into a 2-dimensional problem?

Solution
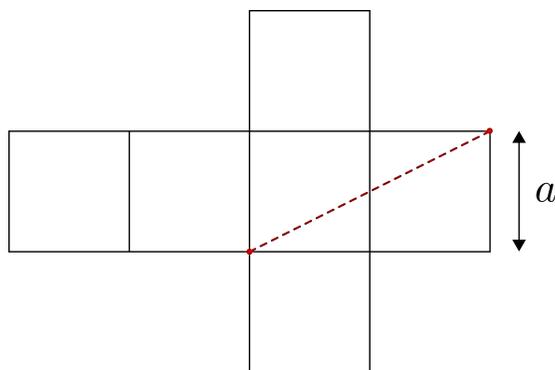
Figure 2.3: A 2D net of a cube showing the minimum distance an ant would have to cover to travel from one corner of the cube to the other.

The shortest path across a 2D surface is a straight line, so unfold the cube into a net as in Figure 2.3. The ant needs to travel the hypotenuse of a

triangle of height a and base $2a$. Let the distance the ant travels be d. By Pythagoras, $d^2 = a^2 + (2a)^2 = 5a^2$ and so our final answer is $d = \sqrt{5}a$.

2.2 Further Problems

7. **Problem**

   Each summer, when the grass grows to a certain height, the grounds-man goes to the uniform cow shop and rents some uniform cows to graze on it, until it reaches a particular level. From the previous 2 years the groundsman knows that it took:

   - Year 1: 6 cows, 4 days to do the job
   - Year 2: 3 cows, 9 days to do the job

   This year only 1 uniform cow is available. How many days will it take for the cow to do the job?

   Hint

   The reason why it doesn't take the 3 cows 8 days to do the job: the grass is growing! Make sure you take this into account. Consider units: what does a cow-day represent physically? What does a cow-day per day represent physically?

   Solution

   Since the grass is growing, our first step should be to work out the daily growth rate of the grass before proceeding with the rest of the question.

   A useful 'unit' to work in is cow-days (very much like the unit of man-hours). This represents a volume of grass (eaten by one cow in one day).

   The initial volume of the grass is the same, but in year 1, 24 cow-days is consumed, whereas in year 2, 27 cow-days is consumed. The daily growth rate is thus the difference in volume divided by the difference in time:

   $$\text{daily growth rate} = \frac{\text{difference in volume}}{\text{difference in time}} \tag{2.5}$$

   $$= \frac{27 \text{ cow-days} - 24 \text{ cow-days}}{9 \text{ days} - 4 \text{ days}} \tag{2.6}$$

   $$= \frac{3}{5} \text{ cow-day per day} \tag{2.7}$$

   This means that a single cow, which eats one cow-day per day, will make inroads into the grass at a rate of 2/5 cow-days per day, taking into account the growth of the grass.

   All that remains is to work out the initial volume of grass. In year 1, 24 cow-days is consumed over 4 days, which means the grass grew by

   $$\frac{3}{5} \times 4 = \frac{12}{5} \text{ cow-days} \tag{2.8}$$

in that time. This means that the initial volume of grass was

$$24 - \frac{12}{5} = \frac{108}{5} \text{ cow-days} \tag{2.9}$$

A similar calculation with year 2 will lead to the same result.

Putting this all together: the initial volume of grass is 21.6 cow-days and a single cow will make inroads into the grass at a rate of 0.4 cow-days per day. The job will therefore be done in $21.6/0.4 = 54$ days.

8. **Problem**
A piranha-infested river runs from west to east, as depicted in Figure 2.4. Sarah Connor is living off the grid 3 km north of the river and does not have access to fresh water. Following a recent impaling, she is unable to walk. Her survival is vital for the future of humanity. Each day Kyle Rees, who lives 2 km north of the river and 12 km west of Sarah Connor, must travel from his hideout with a bucket, which he fills with water from the river. He must save as much energy as possible for the fight against the machines. What is the minimum distance he needs to travel to get to Sarah's house via the river?
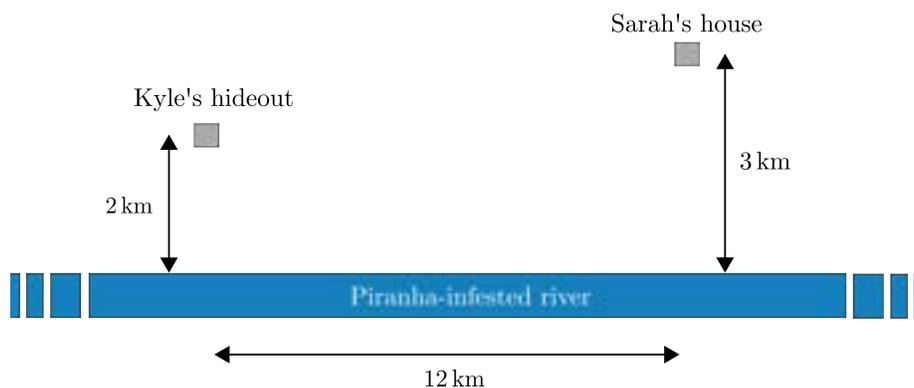
Figure 2.4: Sarah Connor's house and Kyle Rees' hideout

Hint
What is the quickest way between any two points? Can you ask an equivalent question following the path of a ray of light? One approach would be to find the minimum distance using calculus, but there is a much neater solution.

Solution
There are a number of approaches to solving this problem, but a simple one stands out.

- One approach is to pick a variable point along the river and set up an expression for the total distance (using Pythagoras twice) and attempt to minimise this using calculus.
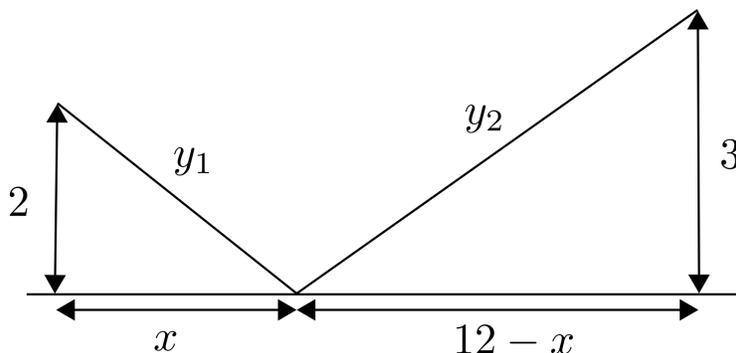
Figure 2.5: Kyle's path to Sarah's house, via the piranha-infested river.

See Figure 2.5. Let the point at which Kyle gets water from the river be a distance x east from his house. Adopting the notation in the diagram, Kyle then travels a total distance $d = y_1 + y_2$ to get to Sarah Connor. Using Pythagoras, this distance is

$$d = y_1 + y_2 = \sqrt{x^2 + 2^2} + \sqrt{(12 - x)^2 + 3^2} \qquad (2.10)$$

We want to minimise this distance with respect to x, and so we need to differentiate d and set the result equal to zero:

$$\frac{d}{dx}(y_1 + y_2) = \frac{2x}{2\sqrt{4 + x^2}} + \frac{2(12 - x)(-1)}{2\sqrt{(12 - x)^2 + 9}} = 0 \qquad (2.11)$$

Rearranging:

$$\frac{x}{\sqrt{4 + x^2}} = \frac{12 - x}{\sqrt{(12 - x)^2 + 9}} \rightarrow \sqrt{\frac{(12 - x)^2 + 9}{4 + x^2}} = \frac{12 - x}{x} \qquad (2.12)$$

Squaring both sides and rearranging to eliminate denominators:

$$x^2(12 - x)^2 + 9x^2 = 4(12 - x)^2 + x^2(12 - x)^2 \qquad (2.13)$$

Cancelling the $x^2(12 - x)^2$ terms on both sides, expanding out and simplifying eventually leads to the quadratic equation

$$5x^2 + 96x - 576 = 0 \qquad (2.14)$$

whose solutions are $x = -24$ and $x = 24/5$. A negative solution to x would be unphysical, so $x = 24/5$. Plugging this into equation 2.10 then gives a distance of $d = 13\,\text{km}$.

- Another approach is to realise that Kyle would follow the path light would take (a straight line) if the river were a mirror and then find similar triangles in a 2:3 ratio.

- However, perhaps the simplest approach is to reflect one of the houses in the river as shown in Figure 2.6. The hypotenuse of the triangle of base $12\,\text{km}$ and height $5\,\text{km}$ is exactly the same distance as that which Kyle has to travel. You may be able to spot that a 5-12-13 Pythagorean triple has been produced! The minimum distance is therefore 13 km.
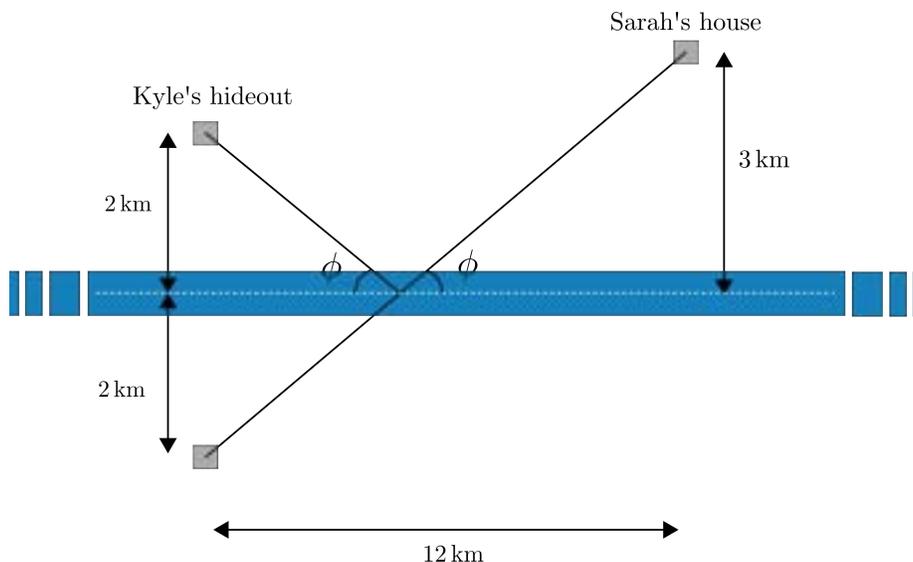

Figure 2.6: A reflection of Kyle's hideout in the river.

9. **Problem**

A person travels from Newcastle to Oxford by coach. Traffic is free-flowing and the coach's speed is only limited by whether the road is flat ($63\,\text{mph}$), uphill ($56\,\text{mph}$) or downhill ($72\,\text{mph}$). The coach ride takes 4 hours from Newcastle to Oxford, but the return journey, which follows the same roads, takes an hour longer. How many miles is the coach ride between Newcastle and Oxford?

Hint

Split the journey into 3 sections of length x, y and z. What happens to the downhill sections on the return journey? What are you trying to find in terms of x, y and z? Do you care what each of them are? The numbers have been chosen carefully – can you tidy up awkward fractions?

Solution

Label the total distances for which the road is uphill, flat or downhill for the outbound journey as x, y and z respectively. This means that on the return journey the flat distance will stay the same, but the uphill and downhill portions will switch. We can then set up two simultaneous

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equations using the fact that time is distance divided by speed:

$$\frac{x}{56} + \frac{y}{63} + \frac{z}{72} = 4 \qquad (2.15)$$

and

$$\frac{z}{56} + \frac{y}{63} + \frac{x}{72} = 5 \qquad (2.16)$$

At first glance it may appear that we have too many unknowns to solve for (3 unknowns but only 2 equations). However, we don't actually care what x, y and z are precisely: we only care about what their sum is. The numbers are chosen nicely: multiplying by 504 (the lowest common multiple of 56, 63 and 72) simplifies things drastically, such that our two simultaneous equations become:

$$9x + 8y + 7z = 4 \times 504 \qquad (2.17)$$

and

$$7x + 8y + 9z = 5 \times 504 \qquad (2.18)$$

Adding these equations together gives

$$16x + 16y + 16z = 16(x + y + z) = 9 \times 504 \qquad (2.19)$$

which means that the total distance is

$$x + y + z = \frac{9 \times 504}{16} = 283.5 \text{ miles} \qquad (2.20)$$

10. **Problem**

It's finally time for a battle to end the long-running dispute between pirates and ninjas. They face off at $90\,\mathrm{m}$. The pirate limps towards the ninja at $2\,\mathrm{m\,s^{-1}}$, while the ninja glides towards the pirate at twice the speed. It is only a matter of time before they collide and crush the loyal parrot which repeatedly flies back and forth at a constant speed of $8\,\mathrm{m\,s^{-1}}$, elastically bouncing off the two. What is the total distance the parrot travels before being crushed?

Hint

One approach would be to find the positions where the parrot changes direction. Since we are given the speed, but asked to find distance, what else would it be useful to know?

Solution

A rigorous way of solving this problem would be to create an infinite geometric sum by finding the positions where the parrot collides, i.e. at $30\,\mathrm{m}$ (having travelled $60\,\mathrm{m}$), then at $66\,\mathrm{m}$ (having travelled $36\,\mathrm{m}$) and so on.

Although it is pretty straightforward to do these calculations by looking at the ratio of the speeds, a simpler way is to pick up on the fact that the parrot's speed is constant at $8\,\mathrm{ms^{-1}}$.

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Since the pirate and ninja get 6 m closer every second, it will take $90/6 = 15$ seconds for them to collide. This means that the total distance the parrot travels before being crushed is $15 \times 8 = 120\,\mathrm{m}$.

2.3 Extension Problems

11. **Tangled resistor circuit**
No calculator
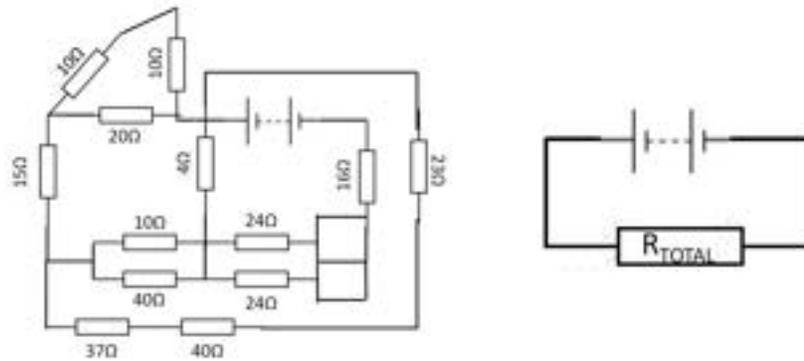If the two circuits in Figure 2.7 shown are equivalent, what is the value of R_{TOTAL}?

Figure 2.7: Two resistor networks

Hint
Redraw the circuit in a more useful/recognizable format, and simplify pairs of resistors as you go. DO NOT try to do it all in one go.

Solution
The key to this problem is to be systematic and methodical. Whilst this may mean drawing a lot of circuit diagrams as you simplify the circuit bit by bit, this is better than trying to do it all in one go and not knowing where you went wrong!

Throughout this solution we will make use of the fact that the total resistance R_S of n resistors in series is

$$R_S = R_1 + R_2 + ... + R_n \tag{2.21}$$

whereas the total resistance R_P of n resistors in parallel is

$$\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n} \tag{2.22}$$

First we will simplify the appearance of the circuit by straightening out any angled wires. This is shown in Figure 2.8.
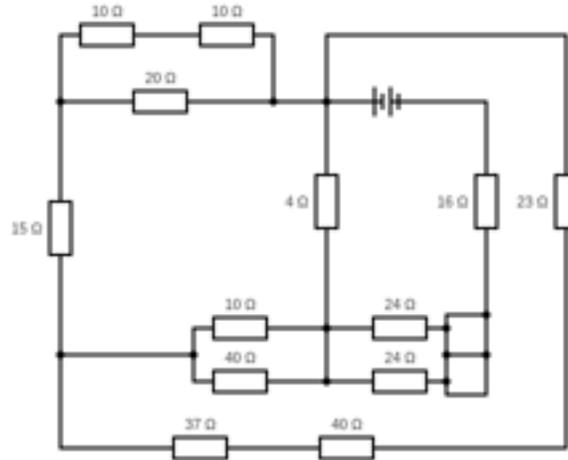
Figure 2.8: Straightening out the angled wires.

Next we will simplify resistors in series by using equation 2.21. The two $10\,\Omega$ resistors in the top left of the circuit combine to form one $20\,\Omega$ resistor, whilst the $37\,\Omega$, $40\,\Omega$ and $23\,\Omega$ resistors on the outside of the circuit combine to form one $100\,\Omega$ resistor. If we also tidy up the mess of wires to the right of the two $24\,\Omega$ resistors, then this results in Figure 2.9.

Now we can combine the two $20\,\Omega$ resistors in the top left of the circuit into one $10\,\Omega$ resistor using equation 2.22. We can also redraw the intersection in the lower middle section of the circuit into an equivalent configuration, which will be helpful later on. This results in Figure 2.10.

This last reconfiguration allows us to combine the two pairs of parallel resistors in the centre of circuit by once again using equation 2.22. The $10\,\Omega$ and $40\,\Omega$ resistors combine to form an $8\,\Omega$ resistor, whilst the two $24\,\Omega$ resistors combine to form a $12\,\Omega$ resistor. We can also merge the $10\,\Omega$ and $25\,\Omega$ resistors in series in the top left of the circuit into one $25\,\Omega$ resistor. This results in Figure 2.11.

The $12\,\Omega$ and $16\,\Omega$ resistors in series can be combined into one $28\,\Omega$ resistor. We will also reposition the $100\,\Omega$ resistor to make it more obvious that it is in parallel with the $25\,\Omega$ resistor. Note that this does not change the circuit at all, as this repositioning is equivalent to repositioning the wires without disconnecting any of the other circuit components. These changes result in Figure 2.12.

Now it is obvious that the $100\,\Omega$ and $25\,\Omega$ resistors can be combined to form a $20\,\Omega$ resistor via equation 2.22. This $20\,\Omega$ resistor will be in series with the $8\,\Omega$ resistor, and so the two can be merged into a $28\,\Omega$ resistor.
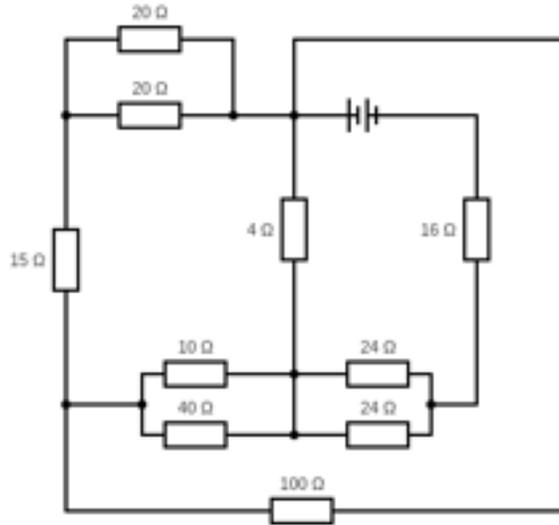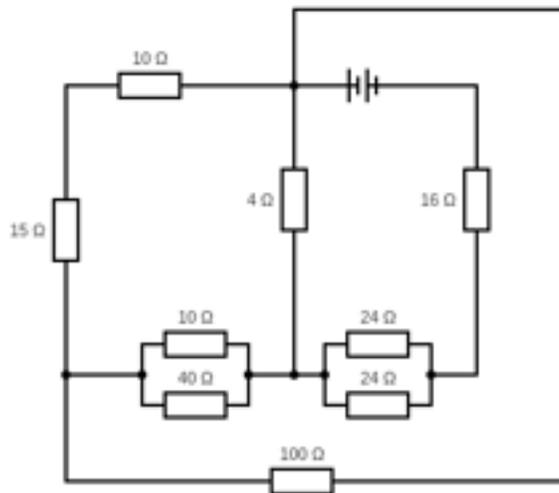
Figure 2.9: Combining resistors in series.

Figure 2.10: Combining resistors in parallel and redrawing the intersection.

The result of this is shown in Figure 2.13.

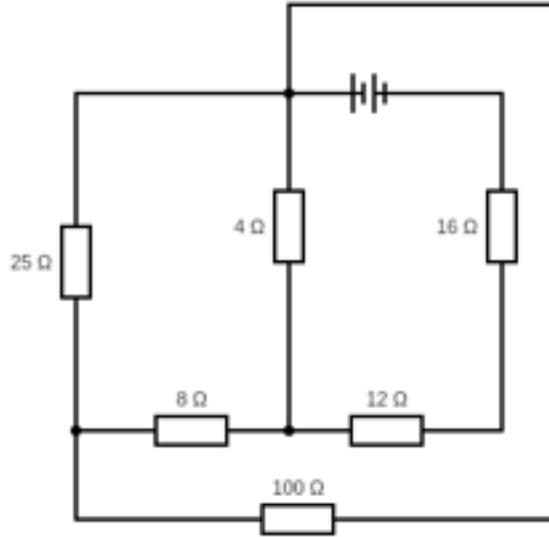The $28\,\Omega$ and $4\,\Omega$ resistors which are in parallel with each other can be

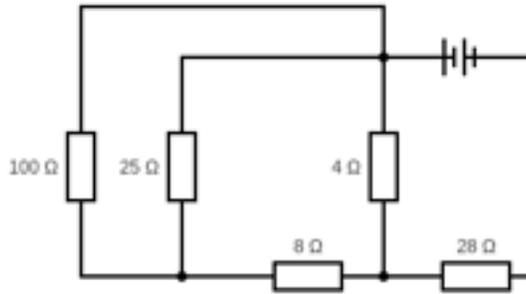Figure 2.11: Combining pairs of resistors in parallel and a pair of resistors in series.

Figure 2.12: Reconfiguring the circuit to make it easier to simplify.

combined to form one $3.5\,\Omega$ resistor. These final two resistors of resistances $3.5\,\Omega$ and $28\,\Omega$ will be in series, and so can be combined via equation 2.21 to result in Figure 2.14.

The final answer is therefore

$$R_{TOTAL} = 31.5\,\Omega \qquad (2.23)$$
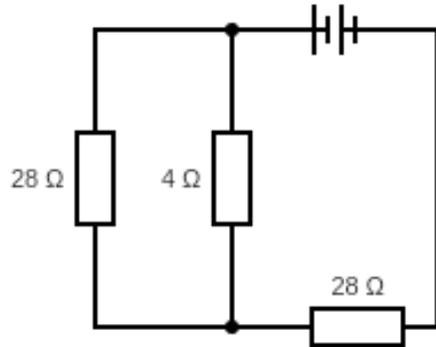
Figure 2.13: Combining a pair of resistors in parallel and then in series.

$$31.5\,\Omega$$
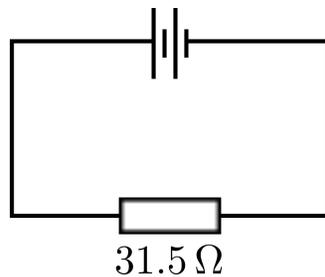
Figure 2.14: The final circuit.

12. **Marble in a Fishbowl**

    No calculator

    A 'fishbowl' of height $\frac{4r}{3}$ is formed by removing the top third of a sphere (of radius r). The fishbowl is fixed in sand so that its rim is parallel with the ground. A small marble of mass m rests at the bottom of the fishbowl. Assuming all surfaces are frictionless and ignoring air resistance, find the maximum initial velocity that could be given to the marble for it to land back in the fishbowl.

    Hint

    Draw a large, clear diagram with all key information labelled. Plan your route through the problem before putting pen to paper. If the initial velocity of the marble is v_0 and the velocity it flies out of the fishbowl is v_1, why is $v_1 < v_0$ if all surfaces are frictionless? Can you solve for the motion of the marble in the air like you would a regular projectile problem? What physical principle will you need to use to relate v_1 to v_0?
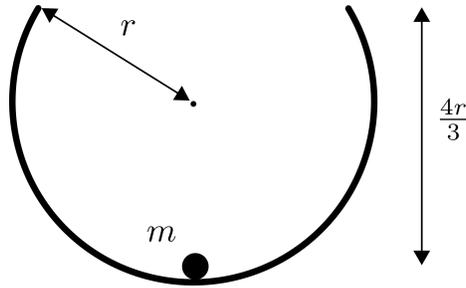
Figure 2.15: A marble at rest at the bottom of a fishbowl.

Solution

Let the velocity given to the marble at the bottom of the fishbowl be v_0. A helpful tactic with involved questions such as these is to think about what happens qualitatively before putting pen to paper. In the case of the marble:

- First it travels with an initial speed of v_0 up the side of the fishbowl

- As it travels up the side of the fishbowl its speed decreases (even though the surface is frictionless) as the marble gains height and converts its initial kinetic energy into gravitational potential energy

- When it reaches the top of the fishbowl it flies off as a projectile at some speed (call this v_1) and some angle (call this θ)

- It then follows a parabolic trajectory and may or may not land back in the fishbowl depending on the value of v_1 and v_0

Our goal is to figure out the value of v_0 such that the marble just about lands back in the fishbowl. The general technique will be to use our standard SUVAT equations when the marble is in the air (as a projectile) and then relate the value of v_1 we obtain to v_0 via the conservation of energy.

First of all, it would be helpful to figure out the horizontal distance s_x which the marble would have to travel in order to land back into the fishbowl. It is worth noting that the angle at which the marble is launched from the fishbowl is identical to the angle from the centre of the sphere to the edge of the fishbowl, as indicated in Figure 2.16. (If this is not immediately obvious, try thinking of limiting cases: what if θ were really small or really large?)

Using the notation adopted in the diagram, we can use Pythagoras to determine the length s_x:

$$r^2 = \left(\frac{s_x}{2}\right)^2 + \left(\frac{r}{3}\right)^2 \rightarrow s_x = \frac{4\sqrt{2}}{3}r \qquad (2.24)$$
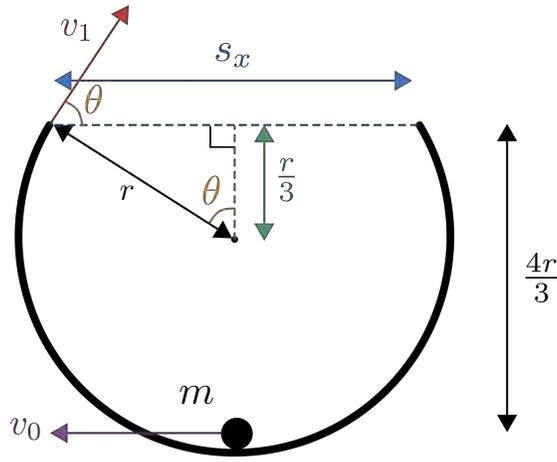
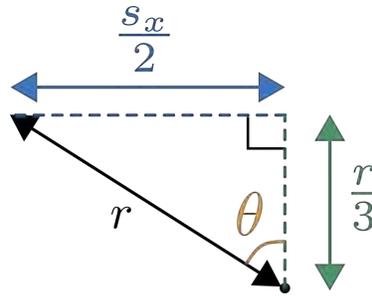Figure 2.16: A marble at rest at the bottom of a fishbowl.

Figure 2.17: A helpful triangle.

It will also be helpful to note from our triangle in Figure 2.17 that

$$\sin \theta = \frac{s_x}{2} \times \frac{1}{r} = \frac{2\sqrt{2}}{3} \qquad (2.25)$$

and

$$\cos \theta = \frac{r}{3} \times \frac{1}{r} = \frac{1}{3} \qquad (2.26)$$

Now let's look at the projectile part of the problem. A good tactic is to lay out the different variables in the horizontal and vertical directions and note down the values which you know. See Table 2.1 for an example of how to do this.

The initial velocities u_i refer to when the marble leaves the fishbowl and the final velocities v_i refer to when the marble lands back into the fishbowl. $s_y = 0$ since the marble will both take off and land at the same vertical level. $u_x = v_x$ since there is no horizontal acceleration such that $a_x = 0$. The time t is the same for both the horizontal and vertical components.

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$$s_x = \frac{4\sqrt{2}}{3}r \qquad s_y = 0$$
$$u_x = v_1 \cos\theta \qquad u_y = v_1 \sin\theta$$
$$v_x = v_1 \cos\theta \qquad v_y = \,?$$
$$a_x = 0 \qquad a_y = -g$$
$$t = \,? \qquad t = \,?$$

Table 2.1: A table of the different variables relevant to the *Marble in a Fishbowl* problem.

Our intermediate goal is to figure out v_1. However, we need to work out t in order to be able to do this. Considering the y direction:

$$s_y = u_y t + \frac{1}{2} a_y t^2 \tag{2.27}$$

Substituting in the relevant values gives

$$0 = (v_1 \sin\theta)t - \frac{g}{2}t^2 \tag{2.28}$$

and so

$$t = \frac{2v_1 \sin\theta}{g} \tag{2.29}$$

Now consider the x direction:

$$s_x = u_x t + \frac{1}{2} a_x t^2 \tag{2.30}$$

Since $a_x = 0$, substituting in values gives

$$\frac{4\sqrt{2}}{3}r = v_1 \cos\theta \times \frac{2v_1 \sin\theta}{g} \tag{2.31}$$

Substituting in the values of $\sin\theta$ and $\cos\theta$ from equations 2.25 and 2.26 and simplifying eventually yields

$$v_1^2 = 3gr \tag{2.32}$$

We are almost there, but not quite! We need v_0, not v_1. Here we need to use the principle of the conservation of energy.

Let the bottom of the fishbowl be the level of zero potential energy. When the marble is at the bottom, its kinetic energy is $mv_0^2/2$ and its potential energy is zero. When the marble flies out of the fishbowl at a speed v_1, its kinetic energy is $mv_1^2/2$ and its potential energy is mgh where h is the height of the fishbowl.

We are told that the height of the fishbowl is $4r/3$. This leads to the energy conservation equation becoming

$$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv_1^2 + mg \times \frac{4r}{3} \tag{2.33}$$

Substituting in $v_1^2 = 3gr$ and simplifying finally yields

$$v_0 = \sqrt{\frac{7gr}{3}} \tag{2.34}$$

13. **Trusses**

No calculator, use $g \approx 10\,\mathrm{ms}^{-2}$

A ball of weight $500\,\mathrm{N}$ is suspended from the apex of the structure shown in Figure 2.18. The structure is made of two trusses, each of length $3.0\,\mathrm{m}$ and mass $40\,\mathrm{kg}$. A $3.0\,\mathrm{m}$ horizontal rope connects the trusses, tied a sixth of the way up the trusses. If the structure were placed on an ice-rink, calculate the resulting tension in the rope.
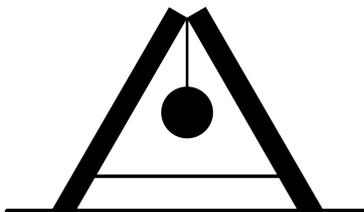
Figure 2.18: Trusses with a suspended ball

Hint

Think about how you might be able to simplify the problem. For example: where might be the best place to take moments about? Make sure you have a clear diagram with all forces labelled.

Solution

A clear diagram with all relevant information is essential to solving this problem well – see Figure 2.19.

The fact that the rope is tied a sixth of the way up the trusses means it is 0.5 m up along the trusses.

By balancing forces in the vertical direction:

$$500 + 2W = 2R \rightarrow R = 250 + W \tag{2.35}$$

We will take moments about the top so that we only need to consider three forces. Having decomposed the respective forces into their directions perpendicular to the trusses, taking moments about the top gives:

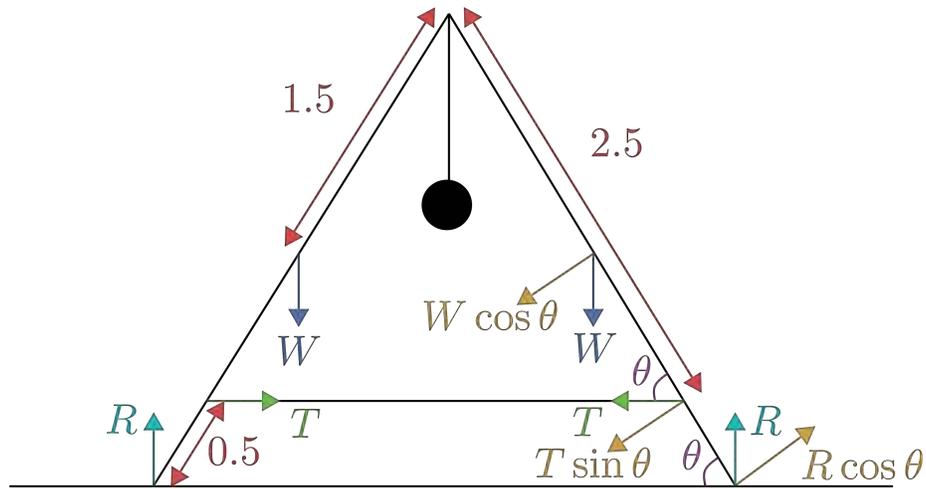$$W\cos\theta \times 1.5 + T\sin\theta \times 2.5 = R\cos\theta \times 3 \tag{2.36}$$

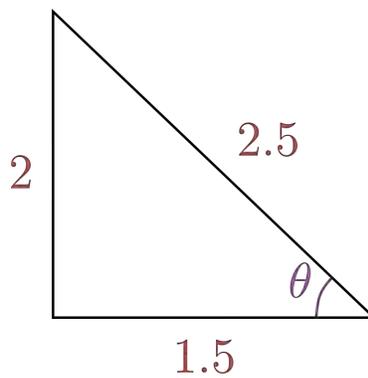Figure 2.19: Trusses with a suspended ball with relevant forces and distances labelled.

Figure 2.20: A helpful triangle.

Multiplying the equation by 2 and rearranging:

$$T \sin \theta = \frac{6R \cos \theta - 3W \cos \theta}{5} \tag{2.37}$$

Substituting the fact that $R = 250 + W$:

$$T = \frac{6(250 + W) \cos \theta - 3W \cos \theta}{5 \sin \theta} \tag{2.38}$$

$$T = \frac{1500 + 3W}{5} \times \frac{\cos \theta}{\sin \theta} \tag{2.39}$$

Using the triangle formed by one of the trusses, the horizontal rope and vertical centre line of the structure (see Figure 2.20), we can use trigonom-

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etry to work out that $\tan\theta = 2/1.5$. Substituting this and the fact that $W = 40g \approx 400$ N:

$$T = \frac{1500 + 3 \times 400}{5} \times \frac{1.5}{2} \tag{2.40}$$

$$T = 405\,\text{N} \tag{2.41}$$

14. **Maximum range**

   (a) Show that the maximum range of a cannon on flat ground is achieved by launching at an angle above the horizontal of $\theta = 45°$.

   (b) Show that, if the cannon is at the top of a hill, with an incline of ϕ, then the range equation can now be written as

   $$R = \frac{u^2}{g}\left[\sin 2\theta + \tan\phi\left(1 + \cos 2\theta\right)\right]. \tag{2.42}$$

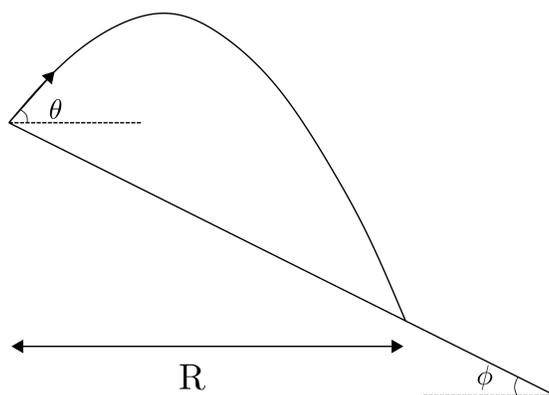   (c) Determine the relationship between θ and ϕ that maximizes the range.

Figure 2.21: The trajectory of a cannonball being fired at an angle θ to the horizontal at the top of a hill with an incline of ϕ.

   Hint

   (a) You'll find it helpful for this particular problem to know that

   $$2\sin x \cos x \equiv \sin 2x \tag{2.43}$$

   In general, follow this step-by-step guide and you'll solve most projectiles problems.

      i. Draw a clear diagram showing all the information given and what you're asked to find.

ii. Split any velocity vectors (usually given the initial velocity) into horizontal and vertical components.

iii. Set up sign convention: which horizontal (x) and vertical (y) directions are positive?

iv. Write out 2 sets of SUVAT, being careful about minus signs.

v. The time of flight will be the same in both directions, so you can often find this by resolving vertically and use this to find the horizontal range.

(b) A clear diagram is always important, more so for tricky problems like this. A stepping stone along the way will be this equation:

$$\frac{1}{2}gt^2 - u\sin\theta t - R\tan\phi = 0. \tag{2.44}$$

You'll find it helpful here to know that

$$\cos 2x \equiv 1 - 2\sin 2x \tag{2.45}$$

(c) Sorry – there's no clever shortcut here, you're going to have to differentiate! You'll find it helpful here to know that

$$\cot\phi = \tan\left(\frac{\pi}{2} - \phi\right) \tag{2.46}$$

Your final expression should be reassuringly simple, and it must resemble what you should have found in part (a) when $\phi = 0$.

Solution

(a) First we will work out the time it takes a projectile to fall back to the ground after being fired, as a function of its launch angle. We will use a standard SUVAT equation:

$$v_y = u_y + a_y t \tag{2.47}$$

Here we can also invoke symmetry: the moment when the projectile is at its highest point (and so $v_y = 0$) is exactly half way along its journey from being fired to landing on the ground. Setting $v_y = 0$, $a_y = -g$ and decomposing the velocity such that $u_y = u\sin\theta$ gives

$$u\sin\theta - gt = 0 \rightarrow t = \frac{u\sin\theta}{g} \tag{2.48}$$

Thus the total time of flight for the projectile is

$$T = 2t = \frac{2u\sin\theta}{g} \tag{2.49}$$

Now, in the x direction

$$s_x = R = u_x T + \frac{1}{2}a_x T^2 \tag{2.50}$$

where $s_x = R$ is the horizontal displacement (or range) of the projectile. Assuming no air resistance, there is no horizontal acceleration, meaning $a_x = 0$ and

$$R = u_x T = u \cos \theta \times \frac{2u \sin \theta}{g} \tag{2.51}$$

where we have substituted for u_x and T. Simplifying:

$$R = \frac{u^2}{g} \times 2 \sin \theta \cos \theta = \frac{u^2}{g} \times \sin 2\theta \tag{2.52}$$

using the trigonometric identity given in the hint. We want to maximise this range R as a function of angle θ. The maximum value of $\sin 2\theta$ is 1 and occurs when $2\theta = 90°$. Therefore the maximum range of a cannon on a flat ground is achieved by launching at an angle $\theta = 45°$.

(b) This is a tough question, so one step at a time! A clear diagram always helps: see Figure 2.22. Using trigonometry we can figure out that the height of the hill must be $R \tan \phi$ where R is the *horizontal* range of the projectile.
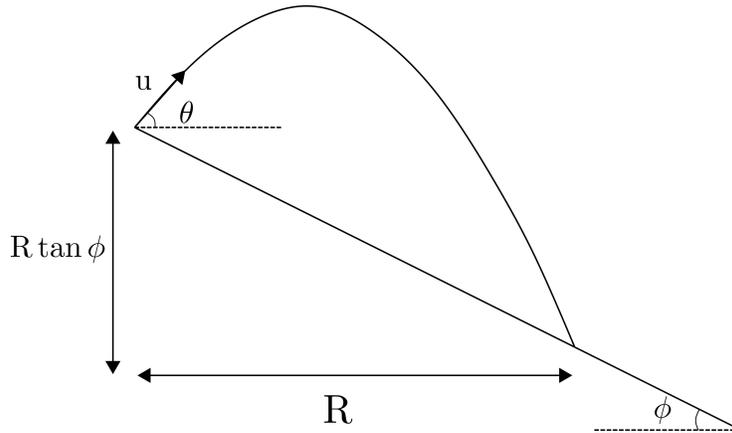

Figure 2.22: The initial trajectory of a cannonball being fired at an angle θ to the horizontal at the top of a hill with an incline of ϕ.

As in previous questions, a useful technique is to lay out all variables (both horizontal and vertical components) to see what you know and what you need to find out. See Table 2.2.

Focusing on the y direction:

$$s_y = u_y t + \frac{1}{2} a_y t^2 \tag{2.53}$$

$$
\begin{aligned}
s_x &= R & s_y &= -R\tan\phi \\
u_x &= u\cos\theta & u_y &= u\sin\theta \\
v_x &= u\cos\theta & v_y &= \,? \\
a_x &= 0 & a_y &= -g \\
t &= \,? & t &= \,?
\end{aligned}
$$

Table 2.2: A table of the different variables relevant to the *Maximum range* problem.

Substituting in values:

$$
-R\tan\phi = (u\sin\theta)t - \frac{g}{2}t^2 \tag{2.54}
$$

and so

$$
\frac{1}{2}gt^2 - (u\sin\theta)t - R\tan\phi = 0 \tag{2.55}
$$

as given in the hint. Using the quadratic formula to solve for t:

$$
t = \frac{u\sin\theta \pm \sqrt{u^2\sin^2\theta + 2gR\tan\phi}}{g} \tag{2.56}
$$

Now focusing on the x direction:

$$
s_x = R = u_x t + \frac{1}{2}a_x t^2 = u_x t \tag{2.57}
$$

since $a_x = 0$. Using $u_x = u\cos\theta$ and the expression for t from above, this yields

$$
R = u_x t = u\cos\theta\left(\frac{u\sin\theta \pm \sqrt{u^2\sin^2\theta + 2gR\tan\phi}}{g}\right) \tag{2.58}
$$

Since we wish to maximise R, we choose the positive square root solution. Now all that is left to do is rearrange for R.

In situations like these, a good tactic is to isolate the square root and then square. Doing this gives:

$$
\left(\frac{Rg}{u\cos\theta} - u\sin\theta\right)^2 = u^2\sin^2\theta + 2gR\tan\phi \tag{2.59}
$$

Expanding out the brackets:

$$
\frac{R^2 g^2}{u^2\cos^2\theta} + u^2\sin^2\theta - \frac{2Rgu\sin\theta}{u\cos\theta} = u^2\sin^2\theta + 2gR\tan\phi \tag{2.60}
$$

Cancelling $u^2 \sin^2 \theta$ from both sides and multiplying everything by $u^2 \cos^2 \theta$:

$$R^2 g^2 - 2Rgu^2 \sin \theta \cos \theta = 2gR \tan \phi u^2 \cos^2 \theta \tag{2.61}$$

Now we use the facts that

$$\sin \theta \cos \theta = \frac{\sin 2\theta}{2} \tag{2.62}$$

and

$$\cos^2 \theta = \frac{\cos 2\theta + 1}{2} \tag{2.63}$$

Plugging these two substitutions into our equation yields

$$R^2 g^2 - Rgu^2 \sin 2\theta = u^2 gR \tan \phi \left[\cos 2\phi + 1\right] \tag{2.64}$$

Finally, dividing by R and rearranging gives

$$R = \frac{u^2 g \tan \phi \left[\cos 2\theta + 1\right] + u^2 g \sin 2\theta}{g^2} \tag{2.65}$$

$$R = \frac{u^2}{g} \left(\sin 2\theta + \tan \phi \left[1 + \cos 2\theta\right]\right) \tag{2.66}$$

This is our final answer.

(c) We need to work out $dR/d\theta$ and set it equal to 0 in order to maximise the range. Note that we do not calculate $dR/d\phi$ and set it equal to 0 as we want to keep the incline of the hill ϕ fixed to find the relationship. Calculating the derivative:

$$\frac{dR}{d\theta} = \frac{u^2}{g} \left(2 \cos 2\theta - 2 \tan \phi \sin 2\theta\right) = 0 \tag{2.67}$$

And so

$$\cos 2\theta = \tan \phi \sin 2\theta \tag{2.68}$$

Rearranging this gives

$$\frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = \frac{1}{\tan \phi} = \cot \phi \tag{2.69}$$

Using the fact that $\cot \phi = \tan (\pi/2 - \phi)$ gives

$$\tan 2\theta = \tan \frac{\pi}{2} - \phi \tag{2.70}$$

and so finally our relationship is

$$2\theta = \frac{\pi}{2} - \phi \tag{2.71}$$

Note that if we set $\phi = 0°$ (such that there is no hill) then we end up with $\theta = 45°$ as before.

15. **Monkey-shoot**

    A zookeeper needs to tranquilise a monkey, who is too shy to come down from the trees, by hitting it with a dart from a tranquiliser gun and catching it in a net as it falls.

    (a) If the monkey does not move, should the zookeeper aim above, at, or below the monkey?

    (b) If the monkey lets go of the branch at the instant the zookeeper shoots the dart, should the zookeeper aim above, at or below the monkey to hit the monkey in mid-air?

    Once you have decided on your answers to this problem, use SUVAT to confirm them.

    Hint
    Figure out the time it takes for the arrow to cross the path of the monkey as a function of the angle which the zookeeper fires the dart. Is the height of the arrow and the monkey the same at this time?

    Solution

    (a) As soon as the dart leaves the zookeeper, it will accelerate downwards due to gravity. Therefore the zookeeper needs to aim above the monkey so that as the dart falls it will hit the monkey.

    (b) Both the monkey and the dart fall at the same rate, regardless of their respective masses (think about the hammer and feather experiment on the Moon). Therefore the zookeeper ought to aim at the monkey to hit it in mid-air.

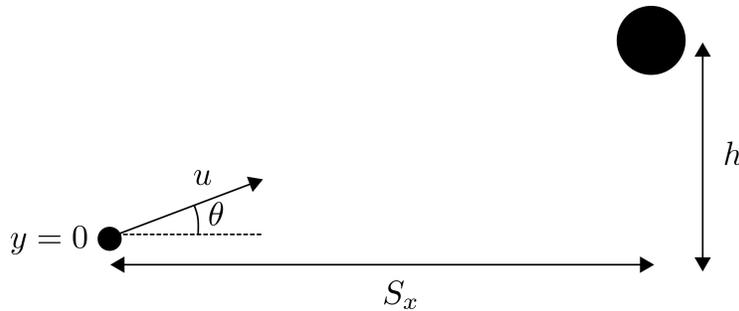   We can confirm part (b) via SUVAT: Figure 2.23 illustrates the situation.

Figure 2.23: A diagram of a dart (small blob) being fired at a monkey (large blob) a distance s_x away at an angle θ to the horizontal.

    Let the horizontal distance between the zookeeper and the monkey be s_x. Let the height where the dart starts be $y = 0$ and the height where the monkey starts be $y = h$.
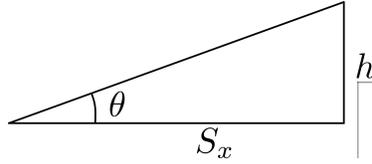
Figure 2.24: A helpful triangle.

Using trigonometry (see Figure 2.24), we can figure out that if the zookeeper aims directly at the monkey, then the angle of the dart's trajectory satisfies

$$\tan \theta = \frac{h}{s_x} \tag{2.72}$$

How long after being fired will the dart reach the monkey? Focusing on the x direction:

$$s_x = u_x t + \frac{1}{2} a_x t^2 \tag{2.73}$$

As usual, assuming no air resistance means that there is no horizontal acceleration and so $a_x = 0$. This means that $s_x = u_x t$ and so

$$t = \frac{s_x}{u_x} = \frac{s_x}{u \cos \theta} \tag{2.74}$$

This is the time it takes the dart to reach the same horizontal displacement as the monkey.

At this time, what are the heights of both the monkey and the dart? For the dart to hit the monkey, these both must be the same.

First the monkey:

$$s_y = u_y t + \frac{1}{2} a_y t^2 + h \tag{2.75}$$

where h is its initial height. $u_y = 0$ since the monkey starts from rest. Plugging in $a_y = -g$ and $t = s_x/(u \cos \theta)$ yields

$$s_y = h - \frac{g s_x^2}{2 u^2 \cos^2 \theta} \tag{2.76}$$

Now the dart:

$$s_y = u_y t + \frac{1}{2} a_y t^2 \tag{2.77}$$

$$= (u \sin \theta) t - \frac{g}{2} t^2 \tag{2.78}$$

$$= u \sin \theta \left(\frac{s_x}{u \cos \theta} \right) - \frac{g}{2} \left(\frac{s_x^2}{u^2 \cos^2 \theta} \right) \tag{2.79}$$

$$= s_x \tan \theta - \frac{g s_x^2}{2 u^2 \cos^2 \theta} \tag{2.80}$$

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Finally, substituting in the fact that if the zookeeper aims directly at the monkey then $\tan\theta = h/s_x$ (equation 2.72), we finally get

$$s_y = h - \frac{gs_x^2}{2u^2\cos^2\theta} \tag{2.81}$$

Since the vertical displacement of both the monkey (equation 2.76) and the dart (equation 2.81) are identical, this means that the zookeeper does indeed need to aim directly at the monkey to hit it in mid-air.

16. **Motion-time graphs**
For the following scenarios, sketch the motion-time graphs (displacement-time, velocity-time and acceleration-time).

(a) A ball is thrown up in the air and then caught at the same height. Only sketch from the instant the ball leaves the hand to the instant it touches the hand again. Ignore air resistance.

(b) A train travels from Birmingham to Oxford, stopping only at Banbury on route. You may assume the journey is in a straight line and it only takes a few minutes of constant acceleration/deceleration to get to its top speed/come to a stop.

(c) A football is dropped from a great height (such that it reaches terminal velocity); it bounces inelastically such that air resistance can be ignored from this point. Sketch the motion from the moment of release until it hits the ground for a second time.

Hint
Your intuition will likely mislead you. Start with a free-body force diagram, link this to acceleration, and then link that to velocity and finally displacement. Split the journey up into regions when the behaviour changes.

Solution

(a) If we ignore air resistance, then the ball's weight is the only force acting on the ball. This means that the acceleration of the ball will be constant (and negative) throughout its motion. This, in turn, means that the velocity-time graph will have a constant, negative gradient, whilst the displacement-time graph will be a parabola which starts at $0\,\text{m}$, reaches some maximum height and then descends back down to $0\,\text{m}$. The graphs are shown in Figure 2.25.

(b) We will first consider the journey from Birmingham to Banbury. The train only accelerates for a short while before reaching its top speed, and only decelerates for a short while before coming to a stop. We can assume that the train reaches its acceleration $a\,\text{m s}^{-2}$ and deceleration $-a\,\text{m s}^{-2}$ almost instantaneously, and that its deceleration is equal in magnitude to its acceleration.
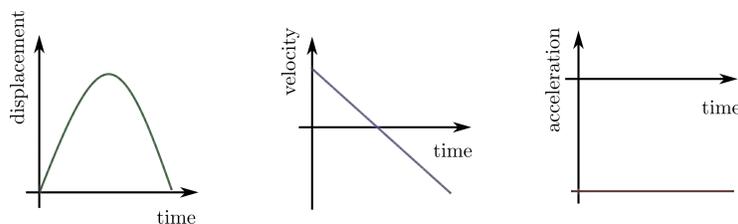
Figure 2.25: Graphs showing the motion of the ball thrown in the air.

The train's velocity-time graph would therefore start at $0\,\mathrm{m\,s^{-1}}$, rise steadily (with a constant gradient) to a maximum velocity of $v\,\mathrm{m\,s^{-1}}$ and then decrease steadily (with a constant gradient) back to $0\,\mathrm{m\,s^{-1}}$. The train's displacement-time graph would start at $0\,\mathrm{m}$, have a constant positive gradient for the period of time that the train is travelling at a constant velocity, and then flatline at a maximum value of $d\,\mathrm{m}$ where d is the distance between Birmingham and Banbury.

Figure 2.26 shows the graphs for the journey from Birmingham to Banbury. The journey from Banbury to Oxford would look exactly the same.

(c) This is a bit more complicated as we want to incorporate the effects of air resistance and inelastic collisions.

First let's consider the acceleration-time graph. As soon as it is dropped, the football accelerates at $-9.81\,\mathrm{m\,s^{-2}}$ due to the Earth's gravitational field. It will then immediately start to feel the effects of air resistance, and so its acceleration downwards will decrease as its resultant force downwards decreases. Once it reaches terminal velocity, the resultant force on the football is zero and so its acceleration is also zero.

When the football bounces inelastically, its acceleration is momentarily upwards and positive as the football changes direction, before returning to $-9.81\,\mathrm{m\,s^{-2}}$ and remaining at this value as air resistance is neglected from this point onwards.

Turning to the velocity-time graph, the football starts at $0\,\mathrm{m\,s^{-1}}$ and its velocity decreases steadily (with a constant gradient) until it starts to feel the effect of air resistance. At this point the gradient becomes less negative, and the velocity eventually levels out at a constant negative value (the football's terminal velocity) and remains at this value until hitting the ground.

When bouncing on the ground its velocity (almost) instantaneously becomes positive as the ball is now travelling upwards. Its velocity then decreases steadily (with a constant gradient), reaching $0\,\mathrm{m\,s^{-1}}$ when the football is at its highest point and becoming more and more negative until hitting the ground for a second time.
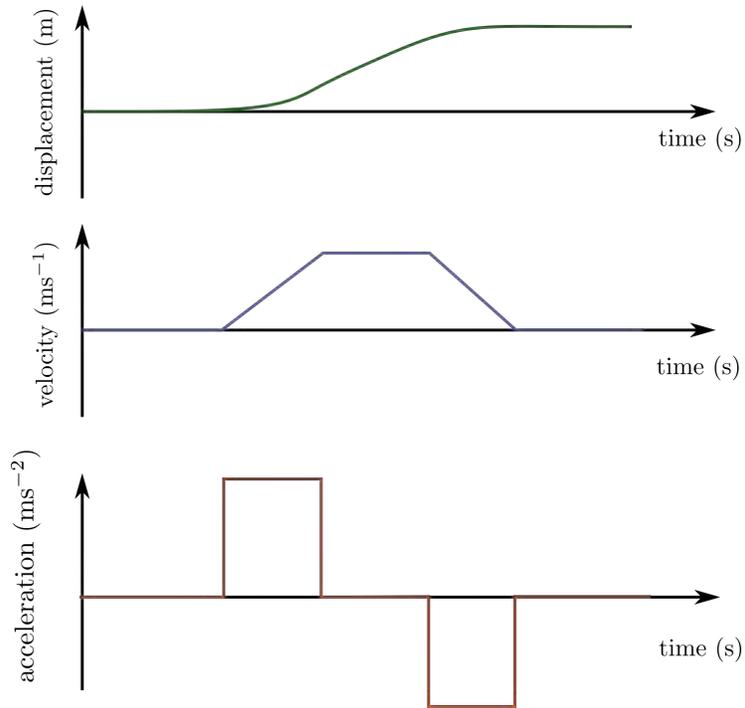
Figure 2.26: Graphs showing the motion of the train from Birmingham to Banbury.

Finally, the displacement-time graph will start at some value h m above the ground, where h is the height from which the ball is dropped. It then decreases non-linearly until the football reaches its terminal velocity, at which point its displacement decreases linearly. Once the football hits the ground, its trajectory is simply a parabola as it is subject to a constant acceleration of $-9.81\,\mathrm{m\,s^{-2}}$ neglecting air resistance.

The graphs are shown in Figure 2.27.

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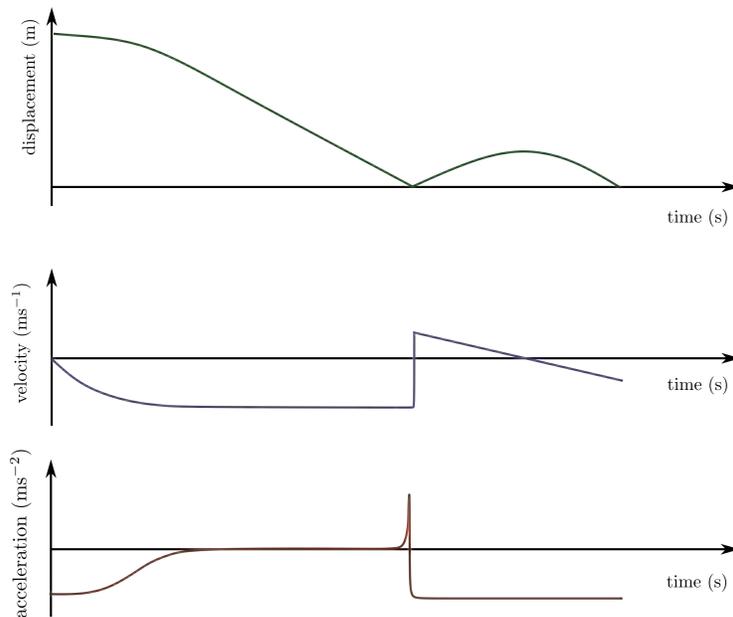
Figure 2.27: Graphs showing the motion of the bouncing ball.

17. **Resistor networks**

A cube of resistors, ABCDEFGH, is made with 12 identical resistors of resistance R, shown in Figure 2.28. A multimeter probe is used to measure its resistance. One wire is connected to vertex A, and the other probe is moved around the other vertices in turn. Determine the resistance reading in each position.
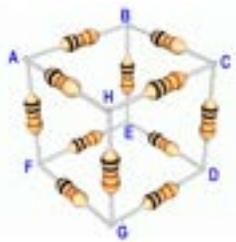

Figure 2.28: A network of 12 identical resistors of resistance R.

Hint

What are the symmetries? Re-draw the network in a more familiar format. No net current will flow between points with the same potential.

Solution

The key idea here will be to use symmetry wherever possible to simplify
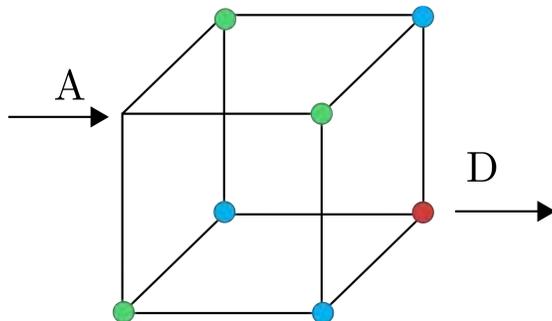
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the problem.

Figure 2.29: A simplified diagram of the resistor cube.

Looking at Figure 2.29, the green points (B, H and F) are all at the same voltage as each other by symmetry, as they are all the same distance from point A. Likewise the blue points (C, G and E) are all at the same voltage as each other as they are all diagonally opposite to A. The red point (D) has a unique voltage.

No current will flow between points which are at the same voltage. This means we can rewrite the circuit as a 2D projection. Put another way, points at the same potential can be connected together by wires and no current will flow, so we can redraw the cube as a circuit as shown in Figure 2.30.
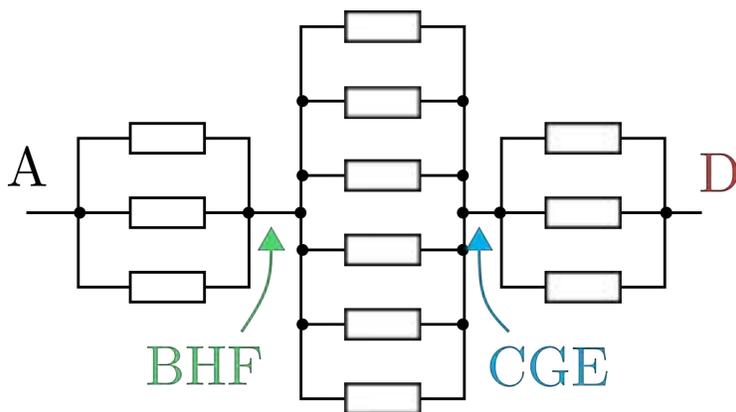

Figure 2.30: A 2D projection of the resistor cube.

Using the standard rules for adding resistors in parallel, this can be simplified to Figure 2.31.

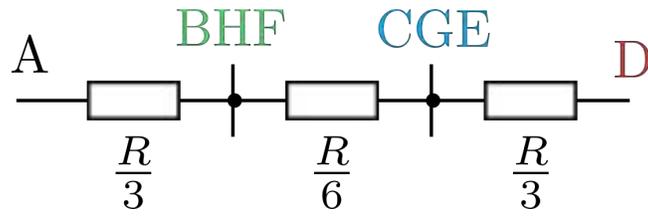The resistances at the various positions are therefore:

Figure 2.31: Simplifying the circuit by combining resistors in parallel.

- B, H and F: $R/3$
- C, G and E: $R/3 + R/6 = R/2$
- D: $R/3 + R/6 + R/3 = 5R/6$