

ADMISSIONS EXERCISE

MSc in Mathematical and Computational Finance

For entry 2020

- *The questions are based on Linear Algebra, Calculus, Probability, Partial Differential Equations, and Algorithms. If you are still studying for a degree and are yet to take or complete courses in these areas, please indicate so here. Please specify the titles and dates of courses which you are due to take/are still taking.*
- *You should attempt all questions and show all working.*
- *Stating the answers without showing how they were obtained will not attract credit.*

Do not turn this page until you are told that you may do so

Statement of authenticity

Please sign and return the following statement together with the solutions. Your application will not be considered without it.

I certify that the work I am submitting here is entirely my own and unaided work.

Print Name _____

Signed _____

Date _____

Probability

1. (a) [3 points] Given a probability space (Ω, \mathcal{F}) and a probability measure \mathbb{P} , state what is a random variable. What is the role of the probability measure \mathbb{P} in the definition of a random variable?
- (b) [3 points] Let Y be a random variable with moment generating function $M(t) = \mathbb{E}(e^{tY})$. Show that

$$\mathbb{P}(Y \geq b) \leq e^{-tb} M(t), \quad \text{for } b > 0, t > 0.$$

- (c) [3 points] Consider a standard normal random variable Z with probability density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty,$$

i.e., $Z \sim N(0, 1)$. Obtain the moment generating function of $\xi = a + bZ$, where a and b are two constants, and Z is a standard normal random variable.

2. [3 points] Let $X \sim N(0, 1)$. Derive the distribution of the random variable X^2 .
3. [3 points] Let X_1, X_2, \dots be a sequence of independently identically distributed random variables with finite mean μ and finite variance. Show that their partial sums $S_n = X_1 + X_2 + \dots + X_n$ satisfy

$$\frac{1}{n} S_n \xrightarrow{D} \mu \quad \text{as } n \rightarrow \infty.$$

4. [3 points] Let X_i be the same as in the previous question with the standard deviation $\sqrt{\text{Var}(X_1)} = \sigma > 0$ and $\mathbb{E}[e^{\lambda X_1}] < +\infty$ for any $\lambda \in \mathbb{R}$. Show that

$$\frac{\sum_{i=1}^n (X_i - \mu)}{\sigma\sqrt{n}} \xrightarrow{D} N(0, 1) \quad \text{as } n \rightarrow \infty.$$

where $N(0, 1)$ means the standard Gaussian distribution.

Statistics

5. A collection of independent random variables X_1, \dots, X_n are modelled with a common distribution defined by

$$P(X_i \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ (x/\beta)^\alpha & \text{if } 0 \leq x \leq \beta \\ 1 & \text{if } x > \beta \end{cases}$$

for fixed positive parameters α, β .

- (a) [3 points] Write down the probability density function of X_i .
- (b) [3 points] Find the maximum likelihood estimators (MLEs) of α and β based on the observations X_1, \dots, X_n .
- (c) [3 points] The length (in mm) of cuckoo's eggs found in hedge sparrow nests can be modelled with this distribution. For the data

22.0, 23.9, 20.9, 23.8, 25.0, 24.0, 21.7,
23.8, 22.8, 23.1, 23.1, 23.5, 23.0, 23.0

evaluate the MLEs of α and β .

- (d) [3 points] Using your estimated values for α and β , and assuming that cuckoo eggs' volumes (in $\text{m}\ell$) satisfy the relationship $V = \frac{3\pi}{32000}L^3$ (due to ellipticity, where L is the egg length in mm), give an estimate for the average volume of a cuckoo's egg, and for the maximum possible volume of a cuckoo's egg.

Analysis

6. (a) [3 points] State Rolle's Theorem for continuous functions on bounded intervals of \mathbb{R} .
- (b) [3 points] Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be such that f has derivatives of all orders and

$$f(x+1) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

Prove that for each $n = 1, 2, \dots$ there exists y_n such that $f^{(n)}(y_n) = 0$.

- (c) Let

$$g(x, y) = (e^x + 1)y^2 + 2(e^{x^2} - e^{2x-1})y + (e^{-x^2} - 1).$$

- (i) [3 points] For any fixed $x \in \mathbb{R}$, show that the equation $g(x, y) = 0$ admits a solution $y(x) \geq 0$, and $\lim_{x \rightarrow 0} y(x) = 0$.
- (ii) [3 points] Show that there exists a constant $\bar{y} > 0$, such that for any fixed $y \in [0, \bar{y}]$, the equation $g(x, y) = 0$ admits a solution $x(y)$.
- (d) A real function f is called convex over $x \in \mathbb{R}$ if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$. A set $\Gamma \subset \mathbb{R}$ is called convex if $\lambda x + (1 - \lambda)y \in \Gamma$ for any $x, y \in \Gamma$ and any $\lambda \in [0, 1]$.
- (i) [3 points] If f is a convex function, show that $\{x : f(x) \leq l\}$ is a convex set for any $l \in \mathbb{R}$.
- (ii) [3 points] If f and g are both convex, show that $h_1(x) := \max\{f(x), g(x)\}$ is convex. Show that $h_2(x) = \min\{f(x), g(x)\}$ can be non-convex.

7. [10 points] Let $p, q \in (1, \infty)$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Prove that for any $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ we have

$$\sum_{k=1}^n |x_k y_k| \leq \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n |y_k|^q \right)^{\frac{1}{q}},$$

i.e.,

$$\|xy\|_1 \leq \|x\|_p \|y\|_q,$$

where $xy = (x_1 y_1, \dots, x_n y_n)$, and for $r \in [1, \infty)$, $\|x\|_r = (\sum_{k=1}^n |x_k|^r)^{\frac{1}{r}}$.

Partial Differential Equations

8. Assume $V(S, t)$ is a smooth function that satisfies the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0, \quad (1)$$

subject to $V(S, T) = \max(S - K, 0)$, $V(0, t) = 0$, $V(S, t) \sim S$ as $S \rightarrow \infty$. Here r , $\sigma \geq 0$, $K \geq 0$ are constants.

The purpose of the exercise is to reduce equation (1), by a suitable change of variables, to the heat equation and then solve for $V(S, t)$.

(a) [3 points] Use the following change of variables

$$t = T - \tau / \frac{1}{2} \sigma^2, \quad S = K e^x, \quad V = K v(x, \tau)$$

to show that (1) becomes

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k - 1) \frac{\partial v}{\partial x} - k v, \quad (2)$$

where $k = r / \frac{1}{2} \sigma^2$ and $v(x, 0) = \max(e^x - 1, 0)$.

(b) [3 points] Now let

$$v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau),$$

for some constants α and β and show that

$$v(x, \tau) = e^{-\frac{1}{2} (k-1) x - \frac{1}{4} (k+1)^2 \tau} u(x, \tau),$$

where

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } -\infty < x < \infty, \tau > 0,$$

with

$$u(x, 0) = \max \left(e^{\frac{1}{2} (k+1) x} - e^{\frac{1}{2} (k-1) x}, 0 \right) \quad (3)$$

and

$$\alpha = -\frac{1}{2} (k - 1), \quad \beta = -\frac{1}{4} (k + 1)^2.$$

(c) [3 points] Show that

$$u(x, \tau) = e^{\frac{1}{2} (k+1) x + \frac{1}{4} (k+1)^2 \tau} \Phi(d_1) - e^{\frac{1}{2} (k-1) x + \frac{1}{4} (k-1)^2 \tau} \Phi(d_2),$$

where $\Phi(y)$ is the normal cumulative density function and

$$d_2 = \frac{\log(S(t)/K) + (r - \frac{1}{2} \sigma^2) (T - t)}{\sigma \sqrt{T - t}}, \quad \text{and} \quad d_1 = d_2 + \sigma \sqrt{T - t}.$$

(d) [3 points] Finally, show that

$$V(S, t) = S \Phi(d_1) - e^{-r(T-t)} K \Phi(d_2). \quad (4)$$

Algebra

9. (a) Consider the system of linear equations $Ax = b$ where A is an $m \times n$ real matrix, and the column vectors x and b are elements in \mathbb{R}^n and \mathbb{R}^m respectively.
- (i) [3 points] Show that Ax lies in the column space of A for any x .
 - (ii) [3 points] Deduce, or prove otherwise, that a solution x exists, for given A and b , if and only if the augmented matrix $(A \mid b)$ has the same rank as A .
 - (iii) [3 points] If $b^\top y = 0$ for any $A^\top y = 0$, show that $b = Ax$ for some vector x .
- (b) [3 points] Let $t \in \mathbb{R}$ and define a matrix A_t by

$$A_t = \begin{pmatrix} 0 & 1 & t \\ 1 & t & 1 \\ t & 1 & 0 \end{pmatrix}.$$

Determine the rank of A_t for any $t \in \mathbb{R}$. Let $b \in \mathbb{R}^3$. For which $t \in \mathbb{R}$ does $A_t x = b$ have a unique solution?

- (c) [3 points] Determine all vectors $b \in \mathbb{R}^3$ such that the system of linear equations $A_0 x = b$ has no solution.
- (d) [3 points] Determine 3×3 invertible matrices P, Q , such that

$$P A_0 Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Algorithms

10. Consider the problem

$$\begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & \cdots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_{n-1} \\ B_n \end{pmatrix}, \quad (5)$$

where n is much greater than one. The matrix is *tridiagonal*, only the elements indicated by a_k , b_k and c_k are *non-zero*. In compact form the problem is

$$\mathbf{A} \mathbf{X} = \mathbf{B},$$

where \mathbf{B} is known and \mathbf{X} must be found. The matrix \mathbf{A} is also diagonally dominant, $|a_k| > 0$, $|c_k| > 0$ and $b_k > |a_k| + |c_k| > 0$, and it has the property that if $d_1 = b_1$ then $d_k = b_k - a_k c_{k-1}/d_{k-1} > 0$ for $k = 2, 3, \dots, n$. These conditions imply that \mathbf{A}^{-1} exists and also that none of the elements of \mathbf{A}^{-1} is zero (you are *not* required to prove either of these two statements).

1. [3 points] Assuming we have \mathbf{A}^{-1} , how many operations does it take to compute \mathbf{X} ? (Only count multiplications and divisions as operations.)
2. [3 points] Show that there is a (unique) LU factorisation

$$\mathbf{A} = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ \ell_2 & d_2 & & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ell_{n-1} & d_{n-1} & 0 \\ 0 & \cdots & 0 & \ell_n & d_n \end{pmatrix} \begin{pmatrix} 1 & u_1 & 0 & \cdots & 0 \\ 0 & 1 & u_2 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & 1 & u_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

by explicitly calculating the ℓ_k , d_k and u_k in terms of the a_k , b_k and c_k . How many operations does it take to compute this factorisation?

3. [3 points] Write the factorisation as $\mathbf{A} = \mathbf{L} \mathbf{U}$ and (5) as $\mathbf{L} \mathbf{U} \mathbf{X} = \mathbf{B}$. Show that it is possible to find \mathbf{X} in $\mathcal{O}(n)$ operations, by solving the problem in two stages,

$$\mathbf{L} \mathbf{V} = \mathbf{B}, \quad \mathbf{U} \mathbf{X} = \mathbf{V}.$$

4. [3 points] Use the previous results to show that it is possible to compute \mathbf{A}^{-1} using $\mathcal{O}(n^2)$ operations, but that it is still always more efficient to use the LU algorithm, described in (2) and (3), to solve problem (5) rather than matrix inversion.