ADMISSIONS EXERCISE

MSc in Mathematical and Computational Finance

For entry 2019

- ***The questions are based on Linear Algebra, Calculus, Probability, Partial Differential Equations, and Algorithms. If you are still studying for a degree and are yet to take or complete courses in these areas, please indicate so here. Please specify the titles and dates of courses which you are due to take/are still taking.***
- *You should attempt all questions and show all working.*
- *Stating the answers without showing how they were obtained will not attract credit.*

Do not turn this page until you are told that you may do so

Statement of authenticity

Please sign and return the following statement together with the solutions. Your application will not be considered without it.

I certify that the work I am submitting here is entirely my own and unaided work.

Print Name ____________________

Signed ____________________

Date ____________________

Probability

1. (a) Let X be a random variable that takes only non-negative values. Show that

$$\mathrm{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}, \quad \text{for } a > 0\,,$$

where $\mathbb{E}[\,\cdot\,]$ is the expectation operator.

(b) Let Y be a random variable with moment generating function $M(t) = \mathbb{E}\big(e^{tY}\big)$. Show that

$$\mathrm{P}(Y \geq b) \leq e^{-t\,b}\,M(t)\,, \quad \text{for } b > 0,\ t > 0\,.$$

(c) Consider a standard normal random variable Z with probability density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty\,,$$

i.e., $Z \sim N(0,1)$. Obtain the moment generating function of $\xi = a + b\,Z$, where a and b are two constants, and Z is a standard normal random variable.

2. Let $X \sim N(0,1)$. Derive the distribution of the random variable X^2.

3. Let $X_1, X_2, \ldots$ be a sequence of independently identically distributed random variables with finite mean μ and finite variance. Show that their partial sums $S_n = X_1 + X_2 + \cdots + X_n$ satisfy

$$\frac{1}{n}\,S_n \xrightarrow{D} \mu \qquad \text{as } n \longrightarrow \infty\,.$$

Statistics

4. A collection of independent random variables $X_1, ..., X_n$ are modelled with a common distribution defined by

$$P(X_i \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ (x/\beta)^\alpha & \text{if } 0 \leq x \leq \beta \\ 1 & \text{if } x > \beta \end{cases}$$

for fixed positive parameters α, β.

(a) Write down the probability density function of X_i.

(b) Find the maximum likelihood estimators (MLEs) of α and β based on the observations $X_1, ., ., X_n$.

(c) The length (in mm) of cuckoo's eggs found in hedge sparrow nests can be modelled with this distribution. For the data

22.0, 23.9, 20.9, 23.8, 25.0, 24.0, 21.7,
23.8, 22.8, 23.1, 23.1, 23.5, 23.0, 23.0

evaluate the MLEs of α and β.

(d) Using your estimated values for α and β, and assuming that cuckoo eggs' volumes (in mℓ) satisfy the relationship $V = \frac{3\pi}{32000} L^3$ (due to ellipticity, where L is the egg length in mm), give an estimate for the average volume of a cuckoo's egg, and for the maximum possible volume of a cuckoo's egg.

Analysis

5. (a) State Rolle's Theorem for continuous functions on bounded intervals of $\mathbb{R}$.

(b) Let $f \colon \mathbb{R} \to \mathbb{R}$ be such that f has derivatives of all orders and

$$f(x+1) = f(x) \qquad \text{for all } x \in \mathbb{R}.$$

Prove that for each $n = 1, 2, \ldots$ there exists y_n such that $f^{(n)}(y_n) = 0$.

(c) Let

$$g(x, y) = (e^x + 1)\, y^2 + 2\,(e^{x^2} - e^{2x-1})\, y + (e^{-x^2} - 1)\,.$$

(i) For any fixed $x \in \mathbb{R}$, show that the equation $g(x, y) = 0$ admits a solution $y(x) \geq 0$, and $\lim_{x \to 0} y(x) = 0$.

(ii) Show that there exists a constant $\bar{y} > 0$, such that for any fixed $y \in [0, \bar{y}]$, the equation $g(x, y) = 0$ admits a solution $x(y)$.

6. Let $p,\ q \in (1,\infty)$, such that $\frac{1}{p}+\frac{1}{q}=1$. Prove that for any $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$ we have

$$\sum_{k=1}^{n}|x_k\,y_k| \le \left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}|y_k|^q\right)^{\frac{1}{q}},$$

i.e.,

$$\|x\,y\|_1 \le \|x\|_p\,\|y\|_q\,,$$

where $x\,y=(x_1\,y_1\,,\ldots,x_n\,y_n)$, and for $r\in[1,\infty)$, $\|x\|_r=\left(\sum_{k=1}^{n}|x_k|^r\right)^{\frac{1}{r}}$.

Partial Differential Equations

7. Assume $V(S,t)$ is a smooth function that satisfies the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\,S\,\frac{\partial V}{\partial S} - r\,V = 0\,, \tag{1}$$

subject to $V(S,T) = \max(S-K,0)$, $V(0,t)=0$, $V(S,t) \sim S \quad \text{as } S \to \infty$. Here r, $\sigma \geq 0$, $K \geq 0$ are constants.

The purpose of the exercise is to reduce equation (1), by a suitable change of variables, to the heat equation and then solve for $V(S,t)$.

(a) Use the following change of variables

$$t = T - \tau/\frac{1}{2}\sigma^2\,, \qquad S = K\,e^x\,, \qquad V = K\,v(x,\tau)$$

to show that (1) becomes

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\,\frac{\partial v}{\partial x} - k\,v\,, \tag{2}$$

where $k = r/\frac{1}{2}\,\sigma^2$ and $v(x,0) = \max(e^x - 1, 0)$.

(b) Now let

$$v(x,\tau) = e^{\alpha\,x + \beta\,\tau}\,u(x,\tau),$$

for some constants α and β and show that

$$v(x,\tau) = e^{-\frac{1}{2}\,(k-1)\,x - \frac{1}{4}\,(k+1)^2\,\tau}\,u(x,\tau),$$

where

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \text{for} -\infty < x < \infty\,,\ \tau > 0\,,$$

with

$$u(x,0) = \max\left(e^{\frac{1}{2}\,(k+1)\,x} - e^{\frac{1}{2}\,(k-1)\,x}, 0\right) \tag{3}$$

and

$$\alpha = -\frac{1}{2}\,(k-1)\,, \qquad \beta = -\frac{1}{4}\,(k+1)^2\,.$$

(c) Show that

$$u(x,\tau) = e^{\frac{1}{2}\,(k+1)\,x + \frac{1}{4}\,(k+1)^2\,\tau}\,\Phi(d_1) - e^{\frac{1}{2}\,(k-1)\,x + \frac{1}{4}\,(k-1)^2\,\tau}\,\Phi(d_2)\,,$$

where $\Phi(y)$ is the normal cumulative density function and

$$d_2 = \frac{\log(S(t)/K) + (r - \frac{1}{2}\,\sigma^2)\,(T-t)}{\sigma\,\sqrt{T-t}}\,, \qquad \text{and} \qquad d_1 = d_2 + \sigma\,\sqrt{T-t}\,.$$

(d) Finally, show that

$$V(S,t) = S\,\Phi(d_1) - e^{-r\,(T-t)}\,K\,\Phi(d_2)\,. \tag{4}$$

Algebra

8. (a) Consider the system of linear equations $A\,x = b$ where A is an $m \times n$ real matrix, and the column vectors x and b are elements in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Show that $A\,x$ lies in the column space of A for any x. Deduce, or prove otherwise, that a solution x exists, for given A and b, if and only if the augmented matrix $(A \mid b)$ has the same rank as A.

(b) Let $t \in \mathbb{R}$ and define a matrix A_t by

$$A_t = \begin{pmatrix} 0 & 1 & t \\ 1 & t & 1 \\ t & 1 & 0 \end{pmatrix}.$$

Determine the rank of A_t for any $t \in \mathbb{R}$. Let $b \in \mathbb{R}^3$. For which $t \in \mathbb{R}$ does $A_t\,x = b$ have a unique solution?

(c) Determine all vectors $b \in \mathbb{R}^3$ such that the system of linear equations $A_0\,x = b$ has no solution.

(d) Determine 3×3 invertible matrices P, Q, such that

$$P\,A_0\,Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$