ADMISSIONS EXERCISE

MSc in Mathematical Finance

For entry 2019

- ***The questions are based on Linear Algebra, Calculus, Probability and Partial Differential Equations. If you are still studying for a degree and are yet to take or complete courses in these areas, please indicate so here. Please specify the titles and dates of courses which you are due to take/are still taking.***
- *You should attempt all questions and show all working.*
- *Stating the answers without showing how they were obtained will not attract credit.*

Statement of authenticity

Please sign and return the following statement together with the solutions. Your application will not be considered without it.

I certify that the work I am submitting here is entirely my own and unaided work.

Print Name ______________________________

Signed ______________________________

Date ______________________________

Probability

1. (a) Let X be a random variable that takes only non-negative values. Show that

$$P(X \geq a) \leq \frac{E(X)}{a}, \quad \text{for } a > 0.$$

(b) Let Y be a random variable with moment generating function $M(t) = E\big(e^{tY}\big)$. Show that

$$P(Y \geq b) \leq e^{-tb} M(t), \quad \text{for } b > 0,\ t > 0.$$

(c) Consider a standard normal random variable Z with probability density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty.$$

Obtain the moment generating function of Z. Hence obtain an upper bound on $P(Z \geq a)$ as a function of t. By optimising over t show that

$$P(Z \geq a) \leq e^{-\frac{1}{2}a^2}, \quad \text{for } a > 0.$$

Statistics

2. A collection of independent random variables $X_1, ..., X_n$ are modelled with a common distribution defined by

$$P(X_i \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ (x/\beta)^\alpha & \text{if } 0 \leq x \leq \beta \\ 1 & \text{if } x > \beta \end{cases}$$

for fixed positive parameters α, β.

(a) Write down the probability density function of X_i

(b) Find the maximum likelihood estimators (MLEs) of α and β based on the observations $X_1, ., ., X_n$.

(c) The length (in mm) of cuckoo's eggs found in hedge sparrow nests can be modelled with this distribution. For the data

22.0, 23.9, 20.9, 23.8, 25.0, 24.0, 21.7,
23.8, 22.8, 23.1, 23.1, 23.5, 23.0, 23.0

evaluate the MLEs of α and β.

(d) Using your estimated values for α and β, and assuming that cuckoo eggs' volumes (in mℓ) satisfy the relationship $V = \frac{3\pi}{32000} L^3$ (due to ellipticity, where L is the egg length in mm), give an estimate for the average volume of a cuckoo's egg, and for the maximum possible volume of a cuckoo's egg.

Analysis

3. (a) Carefully state Rolle's Theorem for continuous functions on bounded intervals of $\mathbb{R}$.

(b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that f has derivatives of all orders and

$$f(x+1) = f(x) \qquad \text{for all } x \in \mathbb{R}.$$

Prove that for each $n = 1, 2, \ldots$ there exists y_n such that $f^{(n)}(y_n) = 0$.

(c) Let

$$g(x,y) = (e^x + 1)y^2 + 2(e^{x^2} - e^{2x-1})y + (e^{-x^2} - 1).$$

(i) For any fixed $x \in \mathbb{R}$, show that the equation $g(x,y) = 0$ admits a solution $y(x) \geq 0$, and $\lim_{x \to 0} y(x) = 0$.

(ii) Show that there exists a constant $\bar{y} > 0$, such that for any fixed $y \in [0, \bar{y}]$, the equation $g(x,y) = 0$ admits a solution $x(y)$.

Partial Differential Equations

4. Consider the initial boundary value problem for the temperature $T(x,t)$ in a rod of length L given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for } 0 < x < L,\ t > 0,$$

with the boundary conditions $T(0,t) = 0$ and $T(L,t) = 0$ for $t > 0$ and the given initial condition $T(x,0) = f(x)$ for $0 < x < L$, where the thermal diffusivity κ is a positive constant.

(a) Use the method of separation of variables to show that, if $T(x,t) = F(x)G(t)$ is a nontrivial solution of the heat equation satisfying the boundary conditions, then for some constant λ

$$F'' = \lambda F \quad \text{for } 0 < x < L \quad \text{with } F(0) = 0, F(L) = 0.$$

By considering the cases (i) $\lambda = -\omega^2$, (ii) $\lambda = 0$ and (iii) $\lambda = \omega^2$, where $\omega > 0$ without loss of generality, determine all real values of λ for which there is a nontrivial solution of the boundary value problem for F and the corresponding separable solutions for T.

(b) Show that for any constants $b_1, b_2, \cdots$, the function

$$T(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right)$$

is a solution to the heat equation. Assuming that the orders of summation and integration may be interchanged, derive integral expressions over $[0, L]$ for the constants b_n for which the general series solution satisfies the initial condition.

Algebra

5. (a) Consider the system of linear equations $Ax = b$ where A is an $m \times n$ real matrix, and the column vectors x and b are elements in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Show that Ax lies in the column space of A for any x. Deduce, or prove otherwise, that a solution x exists, for given A and b, if and only if the augmented matrix $(A \mid b)$ has the same rank as A.

(b) Let $t \in \mathbb{R}$ and define a matrix A_t by

$$A_t = \begin{pmatrix} 0 & 1 & t \\ 1 & t & 1 \\ t & 1 & 0 \end{pmatrix}.$$

Determine the rank of A_t for any $t \in \mathbb{R}$. Let $b \in \mathbb{R}^3$. For which $t \in \mathbb{R}$ does $A_t x = b$ have a unique solution?

(c) Determine all vectors $b \in \mathbb{R}^3$ such that the system of linear equations $A_0 x = b$ has no solution.

(d) Determine 3×3 invertible matrices P, Q, such that

$$PA_0Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$