ADMISSIONS EXERCISE

MSc in Mathematical and Computational Finance

For entry 2017

- ***The questions are based on Linear Algebra, Calculus, Probability, Statistics, and Algorithms.***
- *You should attempt all questions and show all working.*
- *Stating the answers without showing how they were obtained will not attract credit.*

Statement of authenticity

Please sign and return the following statement together with the solutions. Your application will not be considered without it.

I certify that the work I am submitting here is entirely my own and unaided work.

Print Name ________________________________

Signed ________________________________

Date ________________________________

Linear Algebra

1. For any polynomial $p(x) = a_0 + a_1x + \cdots + a_kx^k$ and any square matrix A, $p(A)$ is defined as $p(A) = a_0I + a_1A + \cdots + a_kA^k$. Show that if v is any eigenvector of A and $\chi_A(x)$ is the characteristic polynomial of A, then $\chi_A(A)v = 0$. Deduce that if A is diagonalisable then $\chi_A(A)$ is the zero matrix.

2. Let $M = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}$.

(i) Find a diagonal matrix D and an invertible matrix P such that $M = PDP^{-1}$.

(ii) Find at least one cube root of M, by observing that if $D = E^3$ then $M = (PEP^{-1})^3$.

(iii) Express the infinite series $e^M = \sum_{n=0}^{\infty} \frac{1}{n!}M^n$ (where $M^0 = I$) as a 2×2 matrix with entries involving the constant e. (You may assume any general properties of infinite series of matrices that you need.)

Calculus

3. (a) If $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ are differentiable functions and $y(u, v) := F(u) + G(v)$ prove that y is a solution of the differential equation

$$\frac{\partial^2 y}{\partial u \partial v} = 0.$$

(b) If F and G are twice differentiable and

$$y(x, t) := F(x - ct) + G(x + ct),$$

where c is a positive constant, prove that $y(x, t)$ is a solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

4. (a) Let $\alpha > 1$. Prove that the function

$$f(x) := \begin{cases} x^\alpha \sin(1/x) & \text{if } x > 0 \,, \\ 0 & \text{if } x \leq 0 \end{cases}$$

is differentiable for all $x \in \mathbb{R}$ and find its derivative. For which values of α is f' continuous at 0? Justify your assertion briefly. *[Where they are valid, you may use the chain rule and all the algebraic properties of derivatives. You may also assume that for $x > 0$, $\alpha > 1$, x^α is differentiable.]*

(b) Adapt the above example to find an example of a differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that $g'(0) = 1 > 0$, but g is not monotonic increasing in any interval $(0, \delta)$.

5. By using the transformation $x^2 - y^2 = u, xy = v$, or otherwise, evaluate

$$\iint_D (x^2 + y^2) \mathrm{d}x \ \mathrm{d}y,$$

where D is the finite region in the positive quadrant of the (x,y) plane which is bounded by the curves

$$x^2 - y^2 = \pm 1, \qquad xy = \tfrac{1}{2},$$

and the co-ordinate axes.

Probability

6. A list consists of 1000 non-negative numbers. The sum of the entries is 9000 and the sum of the squares of the entries is 91000. Let X represent an entry picked at random from the list. Find the mean of X, the mean of X^2, and the variance of X. Using Markov's inequality, show that the number of entries in the list greater than or equal to 50 is at most 180. What is the corresponding bound from applying Markov's inequality to the random variable X^2? What is the corresponding bound using Chebyshev's inequality?

7. Let $\{X_i\}_{i \geq 1}$ be i.i.d uniform on random variables [0,1]. Let $M_n = \max \{X_1, \ldots X_n\}$.

(a) Show that $M_n \to 1$ in probability as $n \to \infty$.

(b) Show that $n(1 - M_n)$ converges in distribution as $n \to \infty$. What is the limit?

Statistics

8. A researcher wishes to estimate the density ρ of organisms per unit volume in a liquid. She conducts n independent experiments: in experiment $i = 1, \ldots, n$, she takes a fixed volume v_i of liquid and measures the number of organisms X_i in this volume - - she assumes X_i has a Poisson distribution with mean ρv_i. Find the maximum likelihood estimator $\hat{\rho}$ and find the bias of $\hat{\rho}$.

If the total volume taken is fixed, $\sum_{i=1}^{n} v_i = a$ say, show that the variance of $\hat{\rho}$ only depends on $v_1 \ldots, v_n$ via their sum a.

Algorithms

9. This is the pseudo-code of the algorithm `maxValInd` which takes a finite list `L` of real numbers, and returns the pair `(VAL, IND)`, where `VAL` is the value of the maximum entry in `L`, and `IND` is an index of `VAL` in `L`. Note that `IND` need not be unique.

```
function maxValInd(L)
  let IND = 1
  let VAL = L[IND]
  for i = 1 to length(L)
    if L[i] > VAL then
      let IND = i
      let VAL = L[IND]
    end if
  next i
  return (VAL, IND)
end function
```

In this algorithm,

- the function `length(L)` takes a finite list `L`$= [a_1, \ldots, a_k]$ and returns the number of elements in `L`,
- the expression `L[i]` returns the ith element in the finite list `L`$= [a_1, \ldots, a_k]$.

(i) Construct the algorithm `runningMin` that takes a finite list `L`$= [a_1, \ldots, a_k]$, of length k, and returns two lists `M`$= [m_1, \ldots, m_k]$ and `I`$= [i_1, \ldots, i_k]$, such that $m_j = \min_{1 \le i \le j} a_i$, moreover i_j satisfies $1 \le i_j \le j$ and $a_{i_j} = m_j$. Ensure that `runningMin` completes the search in at most $c_1 k$ steps, for some positive c_1 that does not depend on `L`.

(ii) Briefly explain why your implementation of `runningMin` completes in at most $c_1 k$ steps (where k denotes the length of the input list `L`).

(iii) By using `maxValInd` and `runningMin` or otherwise, construct the algorithm `maxIncrement` that takes a finite list `L`$= [a_1, \ldots, a_k]$ and returns a pair of indices `(IND1, IND2)`, such that $1 \le$ `IND1`$\le$ `IND2`$\le k$, and

$$a_{\texttt{IND2}} - a_{\texttt{IND1}} = \max_{1 \le i \le j \le k} (a_j - a_i).$$

Ensure that `maxIncrement` completes the search in at most $c_2 k$ steps, for some positive constant c_2 that does not depend on `L`.

(iv) Briefly explain why your implementation of `maxIncrement` completes in at most $c_2 k$ steps (where k denotes the length of the input list `L`).